§ Problem Statement

Each point in the plane is labeled with a real number. Prove that there exist two distinct points P and Q such that their labels differ by less than the distance from P to Q.

§ Solution

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the labeling, and suppose the difference in labels for any points $P, Q \in \mathbb{R}^2$ is at least their distance. Then the preimage of any length ℓ interval must lie in a width ℓ square, which has area ℓ^2. However, the preimage of k length ℓ/k intervals lies in an area $k(\ell/k)^2$ region, which tends to 0 as $k \to \infty$. Hence the preimage of any interval can be contained in any region with arbitrarily small area. Therefore, the preimage of $\mathbb{R}$ is contained in the union of countably many regions with arbitrarily small area, contradiction.

§ Variants

Variant A. Let $0 < c < 2$ be a real number and let $f : \mathbb{C} \to \mathbb{R}$ be a function. Prove that there exist complex numbers $z_1, z_2 \in \mathbb{C}$ for which

$$|f(z_2) - f(z_1)| < |z_2 - z_1|^c.$$

Solution sketch. Suppose for the sake of contradiction that there exists a function $f : \mathbb{C} \to \mathbb{R}$ such that

$$|f(z_2) - f(z_1)| > |z_2 - z_1|^c$$

for all complex numbers z_1 and z_2.

By the inequality, the preimage of any length ℓ interval must lie in a width $\ell^{\frac{1}{c}}$ square, which has area $\ell^{\frac{2}{c}}$. However, the preimage of k length ℓ/k intervals lies in an area $k(\ell/k)^{\frac{2}{c}} = \ell^{\frac{2}{c}} k^{1-\frac{2}{c}}$ region, which tends to 0 as $k \to \infty$. Hence the preimage of any interval can be contained in any region with arbitrarily small area.

Therefore, the preimage of $\mathbb{R}$ is contained in the union of countably many regions with arbitrarily small area, contradiction.

Variant B. Find all real numbers c such that for every function $f : \mathbb{C} \to \mathbb{R}$, there exist complex numbers z_1 and z_2 such that

$$|f(z_2) - f(z_1)| < |z_2 - z_1|^c.$$

Solution sketch. It suffices to find a function $f : \mathbb{C} \to \mathbb{R}$ such that

$$|f(z_2) - f(z_1)| > |z_2 - z_1|^c$$

for all complex numbers z_1 and z_2.

Let $g(z) : [0,1] + [0,1]i \to [0,1]$ be the inverse Hilbert curve. The preimage of any interval $\left[\frac{n}{2^{2k}}, \frac{n+1}{2^{2k}}\right]$ is a square of side length $\frac{1}{2^k}$ that is adjacent to the preimage of $\left[\frac{n+1}{2^{2k}}, \frac{n+2}{2^{2k}}\right]$. This means the preimage of any length ℓ interval is contained in a width $4\sqrt{\ell}$ square. This means

$$8\sqrt{|g(z_2) - g(z_1)|} > |z_2 - z_1|,$$

implying that some sufficiently large multiple of $g(z)$, say $h(z)$, satisfies the desired inequality over its domain.

To extend the domain of this solution to all complex numbers, partition the complex plane into countably many unit squares, copy $h(z)$ onto each unit square, and space the images of each unit square sufficiently far apart on the real number line.

§ Comments

I don't know of any other constructions for $c = 2$ besides similar Hilbert-like constructions. I am not sure if fractals are suitable for olympiads, but if they are, feel free to include the variant in the shortlist packet.

§ Metadata

This problem was selected as Problem 2 of the 2023 USEMO.

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- Author: Holden Mui
- Subject: algebra
- Description: inequality involving function from $\mathbb{C}$ to $\mathbb{R}$
- Keywords: complex numbers, functional inequality
- Difficulty: USAMO 1/4 or USAJMO 2/5
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