§ Problem Statement

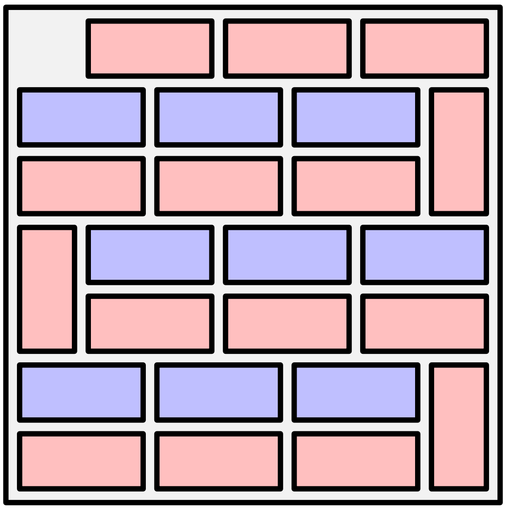
Fix an odd integer n, and place $\frac{1}{2}(n^2-1)$ nonoverlapping dominoes in an $n \times n$ grid. Let k be the number of distinct grid-aligned configurations obtainable by sliding the dominoes. In terms of n, find all possible values of k.

§ Solution

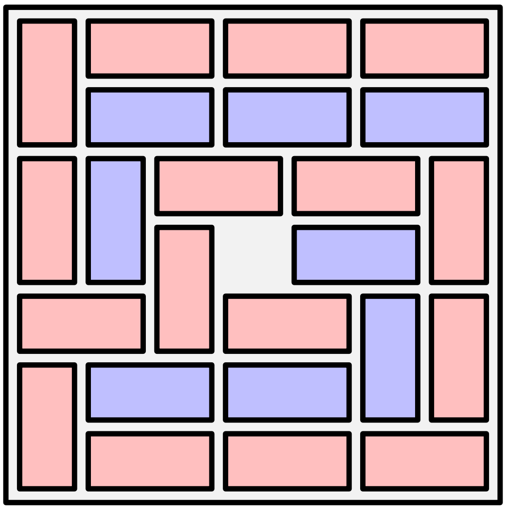
The answer is $k \in \left\{1, 2, \ldots, (\frac{n-1}{2})^2\right\} \cup \left\{(\frac{n+1}{2})^2\right\}$. Call a cell *blue* if its row index and column index are both even, and call a cell *red* if its row index and column index are both odd. Additionally, color each domino with the color of the colored cell it covers.

Constructions

One possible construction for $k = (\frac{n+1}{2})^2$ involves positioning the dominoes covering red cells in a snake-like fashion. An example construction for $n = 7$ is shown below.

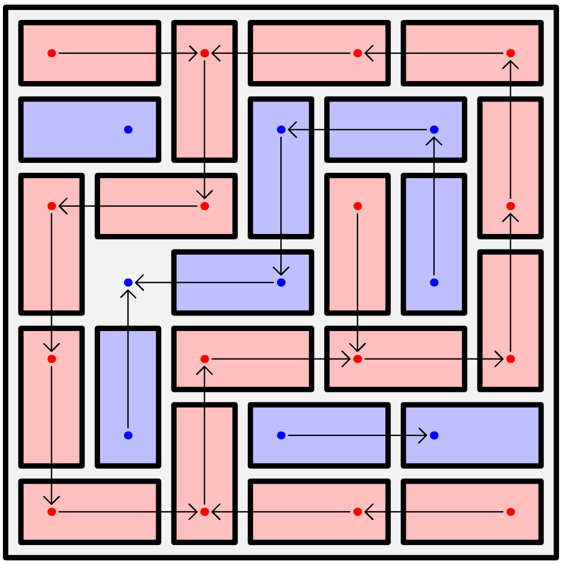


One possible construction for $1 \le k \le (\frac{n-1}{2})^2$ involves positioning the dominoes covering blue cells in a snake-like fashion, blocking the snake's path with a red domino and an empty square, and filling the rest of the grid with red dominoes. An example construction for $n = 7$ and $k = 5$ is given below.



Proof that no other k work

Let G be the directed graph whose vertex set is the red and blue cells, and whose directed edges are drawn from v to the cell the domino covering v points to, if it exists.



Let u denote the uncovered cell. Observe the following:

(i) By a checkerboard coloring argument, u must be a vertex of G.

(ii) Sliding a domino does not change the location of G's edges; it only reverses the direction of an arrow.

(iii) The number of cells in any cycle's interior is odd, so if G contains a cycle, then u is inside the cycle.

(iv) The connected component of G containing u is a tree because it cannot contain any cycles, by (iii).

(v) Given the nondirected edges of G, specifying the uncovered cell is enough to recover the domino configuration, by (iv).

(vi) Hence, k is the number of vertices in the connected component of G containing u.

Because arrows only connect cells of the same color, k is at most the number of red cells, which is $(\frac{n+1}{2})^2$. Therefore, it suffices to prove that $k \notin \{(\frac{k-1}{2})^2 + 1, \ldots, (\frac{k+1}{2})^2 - 1\}$. This follows from the following lemma:

Lemma 1. *If the connected component containing u has more than $(\frac{n-1}{2})^2$ vertices, then it contains every red vertex.*

Proof. If the connected component containing u has more than $(\frac{n-1}{2})^2$ vertices, then u must be red because there are only $(\frac{n-1}{2})^2$ blue vertices. Additionally, at least one vertex in the connected component containing u must border the edge of the grid, so without loss of generality assume u borders the edge of the grid by sliding some of the dominoes.

Now, starting from the red vertex and following the arrows must eventually lead to u or lead to a cycle. By (iii), it is impossible to get into a cycle because u is on the edge. Thus, every red vertex is connected to u, as desired. □

Therefore, $k > (\frac{n-1}{2})^2$ implies that k equals the number of red cells, which is $(\frac{n+1}{2})^2$. This gives the solution set $k \in \left\{1, 2, \ldots, (\frac{n-1}{2})^2\right\} \cup \left\{(\frac{n+1}{2})^2\right\}$, as desired.

§ Variants

Variant A. Fix an odd integer n, and let S be the set of all ways to partition an $n \times n$ grid into $\frac{1}{2}(n^2 - 1)$ dominoes and a 1×1 square. Let G be the graph with vertex set S and an edge connecting two partitions if they differ in the placement of exactly one domino. Determine all possible sizes of the vertex set of a connected component of S.

Solution sketch. This is just a graph-theoretic reformulation of the problem.

Variant B. Fix an odd integer n, and place $\frac{1}{2}(n^2 - 1)$ nonoverlapping dominoes in an $n \times n$ grid. Let k be the number of distinct grid-aligned configurations obtainable by sliding the dominoes. In terms of n, find the largest possible value of k.

Solution sketch. The answer is $k = (\frac{n+1}{2})^2$. The solution to this variant is easier than the solution to the original problem, because it suffices to show that the connected component containing the empty square cannot have any cycles.

§ Comments

I came up with this problem during an Athemath office hours session. There were no students at the time, so I was bored and decided to write a grid combinatorics problem.

§ Metadata

This problem was selected as Problem 3 of the 2023 USAMO and Problem 3 of the 2023 USAJMO.

- Title: Dominoes on a Grid
- Author: Holden Mui
- Subject: combinatorics
- Description: determine the possible number of configurations attainable by moving dominoes on a grid
- Keywords: connected component, domino, graph, grid
- Difficulty: USAMO 2/5 or USAJMO 3/6
- Collaborators: Ankit Bisain, Milan Haiman, Reyna Li, Luke Robitaille
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