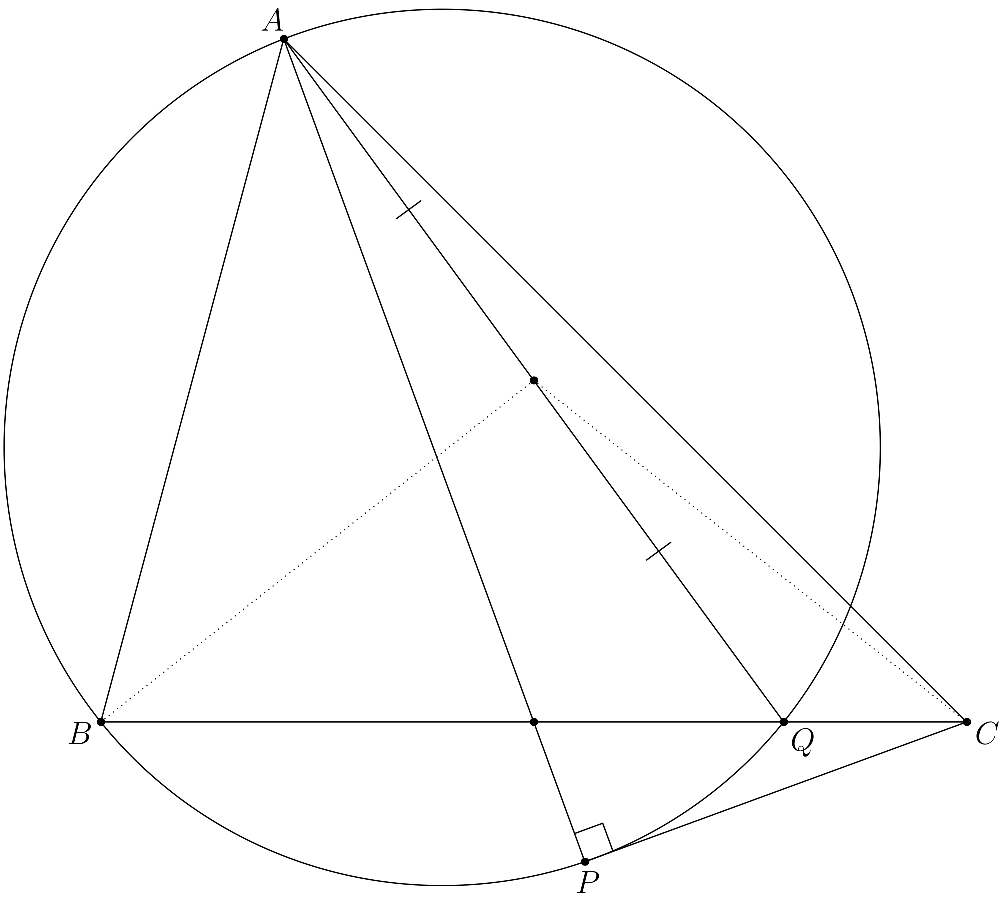
§ Problem Statement

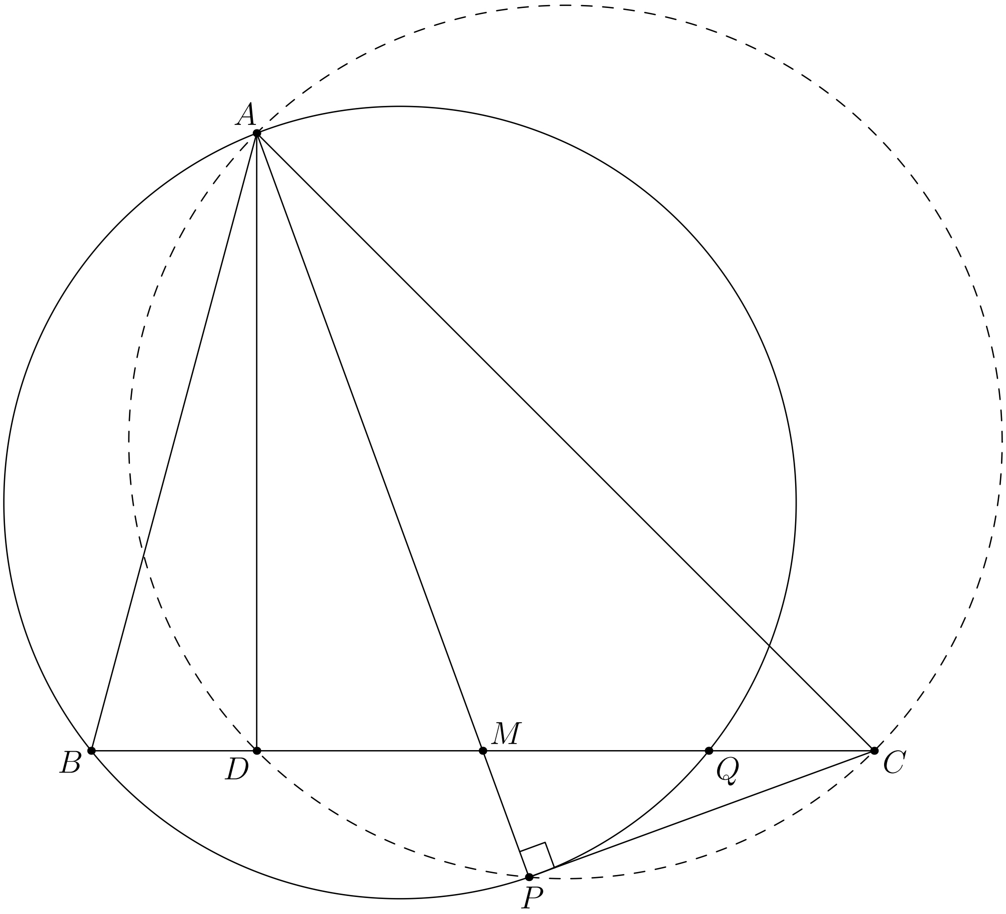
The foot from C to the A-median of triangle ABC is P, and the circumcircle of triangle ABP meets $\overline{BC}$ again at Q. Prove that the midpoint of $\overline{AQ}$ is equidistant from B and C.

§ Diagram



§ Solution

Let D be the foot of the A-altitude and M be the midpoint of $\overline{BC}$. It suffices to show $DM = MQ$.



Note that quadrilateral $ADPC$ is cyclic since $\angle ADC = \angle APC = 90^\circ$. Because $ABM \sim QPM$ and $ACM \sim DPM$ with the same ratio of similitude $\frac{\frac{1}{2}BC}{MP}$,

$$\frac{DM}{CM} = \frac{MQ}{BM} \implies DM = MQ.$$

§ Metadata

This problem was selected as Problem 1 of the 2023 USAMO and Problem 2 of the 2023 USAJMO.

- Title: Foot to Median
- Author: Holden Mui
- Subject: geometry
- Description: midpoint lies on perpendicular bisector in a configuration involving the foot to the median
- Keywords: altitude, foot, median, midpoint, similar triangles
- Difficulty: USAJMO 1/4
- Collaborators: Serena An, Luke Robitaille, Isaac Zhu
- Date written: November 2020
- Submission history: 2022 USAMO, 2023 USAMO