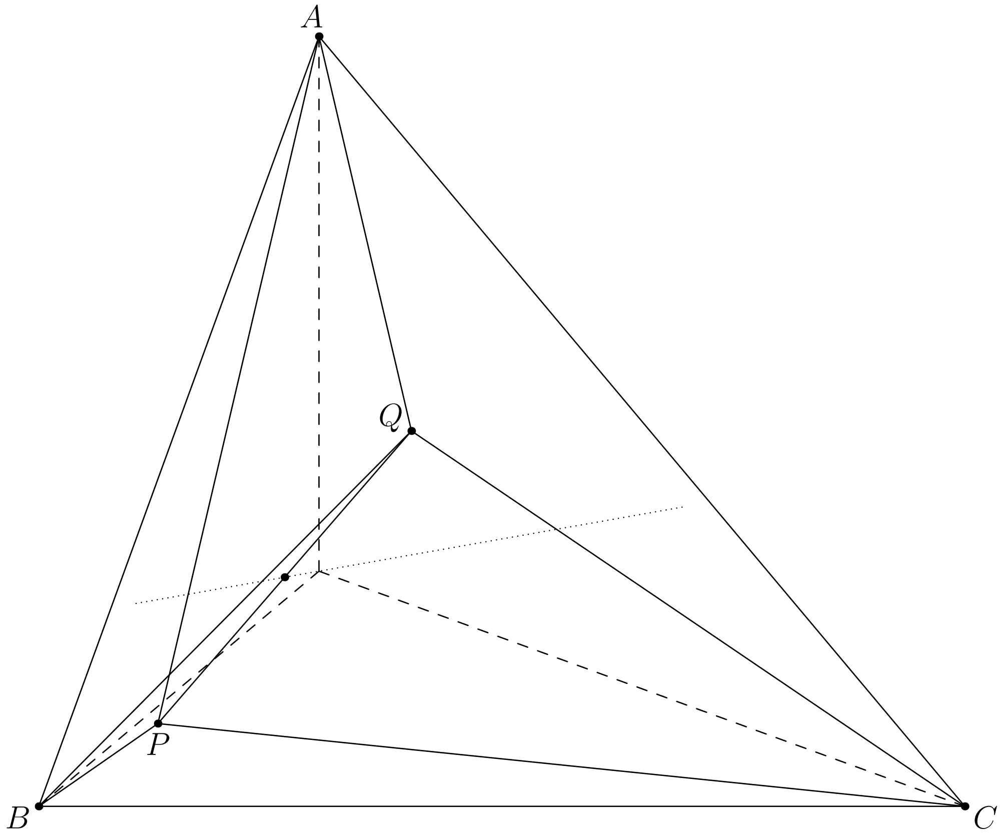
## § Problem Statement

Let distinct points $P$ and $Q$ lie inside scalene triangle $ABC$. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are altitudes of triangle $ABC$. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of triangle $ABC$.

## § Diagram



## § Solutions

### Solution A

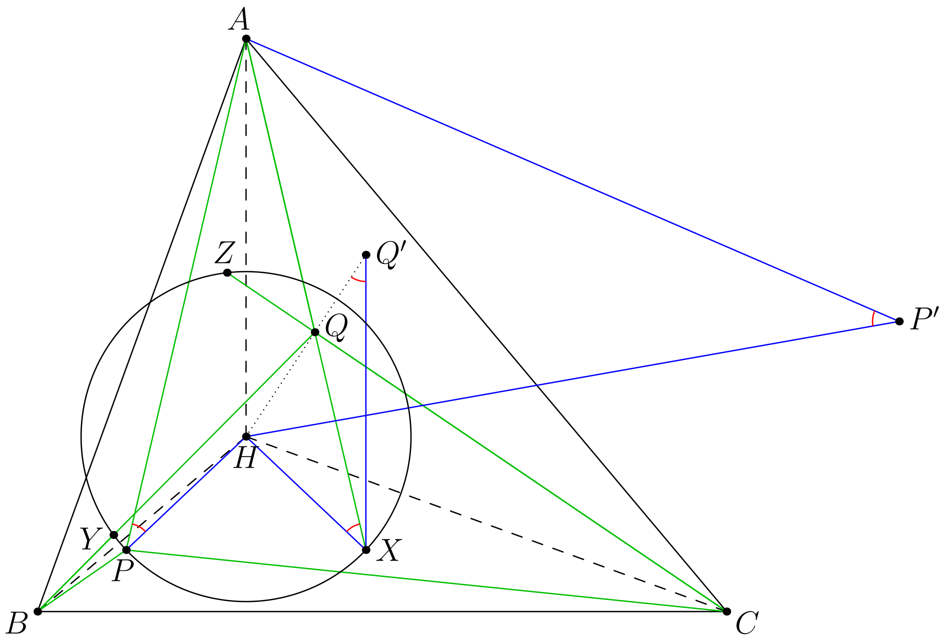
Let $H$ be the orthocenter of $ABC$, and construct $P'$ using the following claim.

**Claim 1.** *There is a point $P'$ for which*

$$\measuredangle APH + \measuredangle AP'H = \measuredangle BPH + \measuredangle BP'H = \measuredangle CPH + \measuredangle CP'H = 0.$$

*Proof.* After inversion at $H$, this is equivalent to the fact that $P$'s image has an isogonal conjugate in $ABC$'s image. □

Now, let $X$, $Y$, and $Z$ be the reflections of $P$ over $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$ respectively. Additionally, let $Q'$ be the image of $Q$ under inversion about $(PXYZ)$.




**Claim 2.** *$ABCP' \sim XYZQ'$.*

*Proof.* Since

$$\measuredangle YXZ = \measuredangle YPZ = \measuredangle(\overline{BH}, \overline{CH}) = \measuredangle BAC$$

and cyclic variants, triangles $ABC$ and $XYZ$ are similar. Additionally,

$$\measuredangle HQ'X = \measuredangle HXQ = \measuredangle HXA = -\measuredangle HPA = \measuredangle HP'A$$

and cyclic variants, so summing in pairs gives $\measuredangle YQ'Z = \measuredangle BP'C$ and cyclic variants; this implies the similarity. □

**Claim 3.** *$Q'$ lies on the Euler line of triangle $XYZ$.*

*Proof.* Let $O$ be the circumcenter of $ABC$ so that $ABCOP' \sim XYZHQ'$. Then $\measuredangle HP'A = \measuredangle HXQ' = \measuredangle OP'A$, so $P'$ lies on $\overline{OH}$. By the similarity, $Q'$ must lie on the $XYZ$'s Euler line. □

To finish the problem, let $G_1$ be the centroid of $ABC$ and $G_2$ be the centroid of $XYZ$. Then

$$\begin{aligned}[G_1HP] + [G_1HQ] &= \frac{[AHP]+[BHP]+[CHP]}{3} + \frac{[AHQ]+[BHQ]+[CHQ]}{3} \\ &= \frac{[AHQ]-[AHX]+[BHQ]-[BHY]+[CHQ]-[CHZ]}{3} \\ &= \frac{[HQX]+[HQY]+[HQZ]}{3} \\ &= [QG_2H] \\ &= 0\end{aligned}$$

where the last line follows from Claim 3. Therefore $\overline{G_1H}$ bisects $\overline{PQ}$, as desired.

## Solution B

Let $(ABC)$ be the unit circle in the complex plane, and let $A = a$, $B = b$, $C = c$ such that $|a| = |b| = |c| = 1$. Let $P = p$ and $Q = q$, and $O = 0$ and $H = h = a + b + c$ be the circumcenter and orthocenter of $ABC$.

The condition that the altitude $\overline{AH}$ bisects $\angle PAQ$ is equivalent to

$$\begin{aligned}&\frac{(p-a)(q-a)}{(h-a)^2} = \frac{(p-a)(q-a)}{(b+c)^2} \in \mathbb{R} \\ \implies &\frac{(p-a)(q-a)}{(b+c)^2} = \overline{\frac{(p-a)(q-a)}{(b+c)^2}} = \frac{(a\overline{p}-1)(a\overline{q}-1)b^2c^2}{(b+c)^2a^2} \\ \implies &a^2(p-a)(q-a) = b^2c^2(a\overline{p}-1)(a\overline{q}-1) \\ \implies &a^2pq - a^2b^2c^2\overline{pq} + (a^4 - b^2c^2) = a^3(p+q) - ab^2c^2(\overline{p}+\overline{q}).\end{aligned}$$

Similarly, the conditions that $\overline{BH}$ and $\overline{CH}$ bisect $\angle PBQ$ and $\angle PCQ$ are equivalent to

$$\begin{aligned}b^2pq - a^2b^2c^2\overline{pq} + (b^4 - c^2a^2) &= b^3(p+q) - bc^2a^2(\overline{p}+\overline{q}) \\ c^2pq - a^2b^2c^2\overline{pq} + (c^4 - a^2b^2) &= c^3(p+q) - ca^2b^2(\overline{p}+\overline{q}).\end{aligned}$$

Now, sum $(b^2 - c^2)$ times the first equation, $(c^2 - a^2)$ times the second equation, and $(a^2 - b^2)$ times the third equation. On the left side, the coefficients of $pq$ and $\overline{pq}$ are 0. Additionally, the coefficient of 1 (the parenthesized terms on the left sides of each equation) sum to 0, since

$$\sum_{\text{cyc}}(a^4 - b^2c^2)(b^2 - c^2) = \sum_{\text{cyc}}(a^4b^2 - b^4c^2 - a^4c^2 + c^4b^2).$$

Therefore,

$$(p+q)\sum_{\text{cyc}} a^3(b^2-c^2) = (\overline{p}+\overline{q})abc\sum_{\text{cyc}}(bc(b^2-c^2)).$$

Consider the cyclic sum on the left as a polynomial in $a$, $b$, and $c$. If $a = b$, then it simplifies as $a^3(a^2-c^2)+a^3(c^2-a^2)+c^3(a^2-a^2) = 0$, so $a-b$ divides this polynomial. Similarly, $a-c$ and $b-c$ divide it, so it can be written as $f(a,b,c)(a-b)(b-c)(c-a)$ for some symmetric quadratic polynomial $f$, and thus it is some linear combination of $a^2+b^2+c^2$ and $ab+bc+ca$. When $a = 0$, the whole expression is $b^2c^2(b-c)$, and so $f(0,b,c) = -bc$, which implies that $f(a,b,c) = -(ab+bc+ca)$.

Now, consider the cyclic sum on the right as a polynomial in $a$, $b$, and $c$. If $a = b$, then it simplifies as

$$ac(a^2-c^2)+ca(c^2-a^2)+a^2(a^2-a^2) = 0,$$

so $a-b$ divides this polynomial. Similarly, $a-c$ and $b-c$ divide it, so it can be written as $g(a,b,c)(a-b)(b-c)(c-a)$ where $g$ is a symmetric linear polynomial; hence, it is a scalar multiple of $a+b+c$. When $a = 0$, the whole expression is $bc(b^2-c^2)$, so $g(0,b,c) = -b-c$, which implies that $g(a,b,c) = -(a+b+c)$.

Finally,

$$(p+q)(ab+bc+ca) = (\overline{p}+\overline{q})abc(a+b+c) \iff (p+q)\overline{h} = (\overline{p}+\overline{q})h$$

since $A$, $B$, and $C$ are distinct. This implies that $\frac{\frac{p+q}{2}-0}{h-0}$ is real, and so the midpoint of $\overline{PQ}$ lies on line $\overline{OH}$.

## Solution C

First, consider the following projective lemma.

**Lemma 4.** *Let $\ell$ be a line, let $\varphi$ be an involution on $\ell$, and call a degree-two polynomial $f \in \mathbb{C}[x,y,z]$* involutive *if $f(P) = 0 \iff f(\varphi(P)) = 0$ for every point $P \in \ell$. Then any linear combination of involutive functions is involutive.*

*Proof.* It suffices to prove the result for two involutive polynomials $f_1$ and $f_2$, since the result will follow via induction. Let $f = \alpha f_1 + \beta f_2$ for some $\alpha, \beta \in \mathbb{C}$.

By Bezout's theorem, the zero sets of $f_1$ and $f_2$ must intersect at four points, which must also lie in $f$'s zero set. Hence Desaurges' involution theorem on $f$, $f_1$, and $f_2$ implies that $f$ must be involutive. □

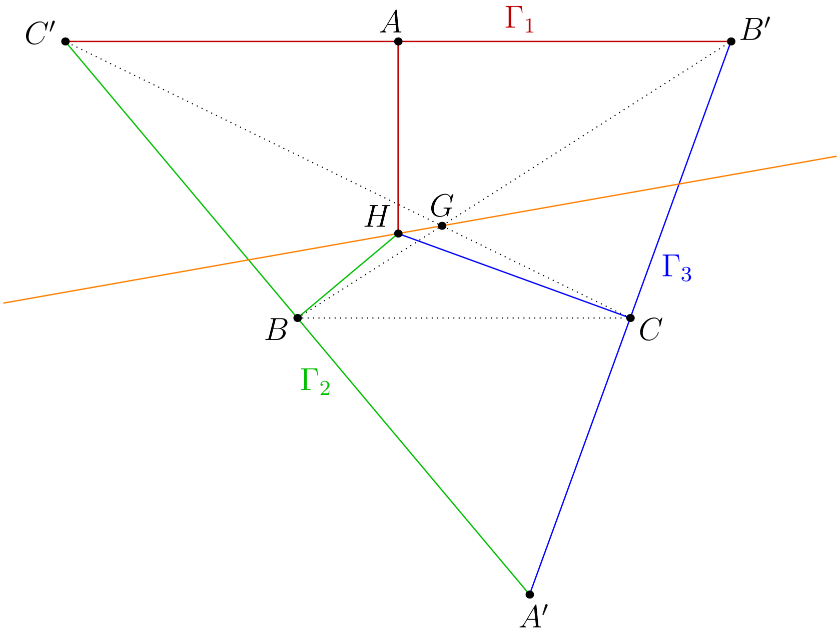
Now, define degree-two polynomials $f_1, f_2, f_3 \in \mathbb{C}[x,y,z]$ such that their zero sets are exactly the conics

$$\Gamma_1 = \overline{AR_1} \cap \overline{AS_1}$$
$$\Gamma_2 = \overline{AR_2} \cap \overline{AS_2}$$
$$\Gamma_3 = \overline{AR_3} \cap \overline{AS_3}.$$

Let $f$ be a nontrivial linear combination of $f_1$, $f_2$, and $f_3$ whose zero set contains some two non-$H$ points on $ABC$'s Euler line; this is possible because there are two degrees of freedom. Since $f(H) = 0$, $ABC$'s Euler line intersects the zero set of $f$ at three distinct points. Thus, Bezout's theorem implies that $f$'s zero set must contain the entire Euler line, so $f$'s zero set is a union of two lines.

**Claim 5.** *$f$'s zero set is the union of $ABC$'s Euler line and the line at infinity.*

*Proof.* Let $A'$ denote the reflection of $A$ over the midpoint of $\overline{BC}$, let $B'$ denote the reflection of $B$ over the midpoint of $\overline{CA}$, and Let $C'$ denote the reflection of $A$ over the midpoint of $\overline{AB}$.

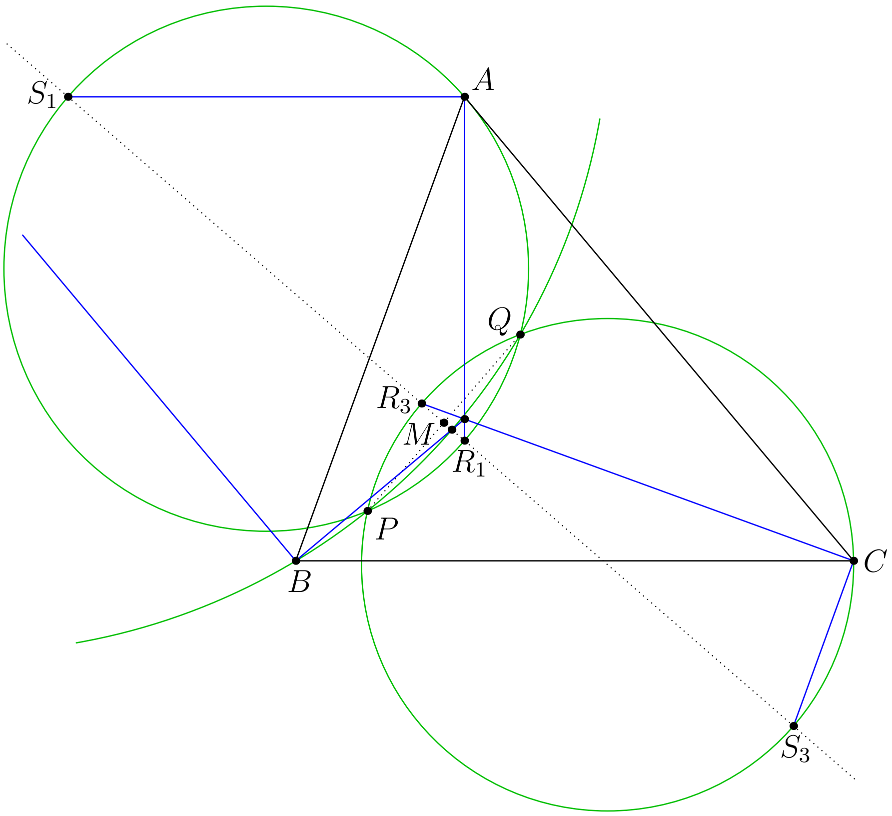



Applying the dual of Desargues' involution theorem to quadrilateral $BCC'B'$ shows that $\{HB, HC'\}$, $\{HC, HB'\}$, $\{HG, HP_\infty\}$ are pairs of some involution (here $P_\infty = \overline{BC} \cap \ell_\infty$). Projecting onto $\overline{B'C'}$ shows that $\{C', \overline{BH} \cap B'C'\}$, $\{B', \overline{CH} \cap B'C'\}$, and $\{P_\infty, \overline{GH} \cap \overline{B'C'}\}$ are pairs of some involution $\varphi$. By applying Lemma 4 on $f_1$, $f_2$, and $f_3$, it follows that $f(\overline{BC} \cap \ell_\infty) = 0$. By symmetry,

$$f(\overline{BC} \cap \ell_\infty) = f(\overline{CA} \cap \ell_\infty) = f(\overline{AB} \cap \ell_\infty) = 0.$$

By Bezout's theorem, $\ell_\infty$ must lie in $f$'s zero set, implying the claim. □

Now, let $M$ be the midpoint of $\overline{PQ}$, and let $\ell$ be the perpendicular bisector of $\overline{PQ}$. Let $\ell$ meet $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$ at $R_1$, $R_2$, and $R_3$. Let $\ell$ meet the line through $A$ perpendicular to $\overline{AH}$, the line through $B$ perpendicular to $\overline{BH}$, the line through $C$ perpendicular to $\overline{CH}$ at $S_1$, $S_2$, and $S_3$.




Observe that $(APQR_1S_1)$ is cyclic with diameter $\overline{R_1S_1}$, $(BPQR_2S_2)$ is cyclic with diameter $\overline{R_2S_2}$, and $(CPQR_3S_3)$ is cyclic with diameter $\overline{R_3S_3}$. By power of a point,

$$MR_1 \cdot MS_1 = MR_2 \cdot MS_2 = MR_3 \cdot MS_3 = PM \cdot PQ = -(\tfrac{1}{2}PQ)^2,$$

so the involution $\varphi : \ell \to \ell$ given by negative inversion with radius $\frac{1}{2}PQ$ swaps the four pairs $\{R_1, S_1\}$, $\{R_2, S_2\}$, $\{R_3, S_3\}$, and $\{M, P_\infty\}$.

Finally, applying Lemma 4 with the pairs $\{R_1, S_1\}$, $\{R_2, S_2\}$, and $\{R_3, S_3\}$, shows that the zero set of $f$ must intersect $\ell$ at $\{M, P_\infty\}$. Therefore the Euler line of $ABC$ must contain $M$, as desired.

## § Comments

Solution A characterizes the set of all points $P$ for which such a point $Q$ exists. Indeed, the set of all such points is the image of the Euler line under the map described in Claim 1.

## § Metadata

This problem was selected as Problem 6 of the 2023 TSTST.

- Title: Euler Line Bisection
- Author: Holden Mui
- Subject: geometry
- Description: midpoint of segment lies on Euler line in five-point problem
- Keywords: angle bisector, midpoint, complex numbers, Euler line
- Difficulty: TSTST 3/6
- Collaborators: Ankit Bisain, Kevin Cong, Ram Goel, Andrew Gu, Luke Robitaille, Carl Schildkraut, Andrew Wu
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- Submission history: 2022 USEMO, 2023 TSTST
- Other credits: the author of Solution A is Ankit Bisain, the author of Solution B is Carl Schildkraut, and the author of Solution C is Juri Kaganskiy.