§ Problem Statement

Let $O = (0, 0)$ and call a lattice triangle ABC *marine* if $[ABO] = [BCO] = [CAO] = \frac{1}{2}$. Find all points in $\mathbb{R}^2$ that don't lie in the interior of any marine triangle.

§ Solution

The answer is $\{(ka, kb) \mid (a, b) \in \mathbb{Z}^2 \setminus O, k \geq 1\}$. All such points cannot lie inside any marine triangle because some side AB of any lattice triangle ABC containing (ka, kb) must intersect the ray starting at (a, b) pointing away from O. This forces $\triangle ABO$ to contain (a, b) so by Pick's theorem,

$$[ABO] \geq \min\Bigg(\underbrace{1 + \frac{1}{2} \cdot 3 - 1}_{\text{in interior}}, \underbrace{0 + \frac{1}{2} \cdot 4 - 1}_{\text{on perimeter}}\Bigg) > \frac{1}{2}.$$

Hence $\triangle ABC$ cannot be marine.

To prove that all other points are contained in some marine triangle, note that the transformations

$$(x, y) \mapsto (x - y, y) \quad \text{and} \quad (x, y) \mapsto (x, y - x)$$

preserve the marine-ness of triangles since shear transformations preserve area. Therefore (x, y) lies in a marine triangle if its image under one of the above transformations lies in a marine triangle.

Assume (x, y) has nonnegative coordinates; the other cases follow by symmetry. By repeatedly subtracting the smaller coordinate from the larger coordinate, (x, y) can be mapped to a point with sufficiently small positive coordinates, or to the point $(k, 0)$ if $(x, y) = (ka, kb)$ for some $(a, b) \in \mathbb{Z}^2_{\geq 0} \setminus O$. If $k < 1$ then the point lies inside the marine triangle with vertices $(1, 0), (0, 1), (-1, -1)$ as desired.

§ Variants

Variant A. Let $O = (0, 0)$ and call a lattice triangle ABC *marine* if O lies inside ABC and $[ABO] = [BCO] = [CAO] = \frac{1}{2}$. Find all points in $\mathbb{R}^2$ that don't lie in the interior of any marine triangle.

Solution sketch. The solution is the same as the solution to the original formulation.

§ Comments

There is another solution that uses rational convergents, but it is essentially isomorphic to the solution given above.

§ Metadata

This problem was selected as Problem 4 of the 2023 Math Prize for Girls Olympiad.

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