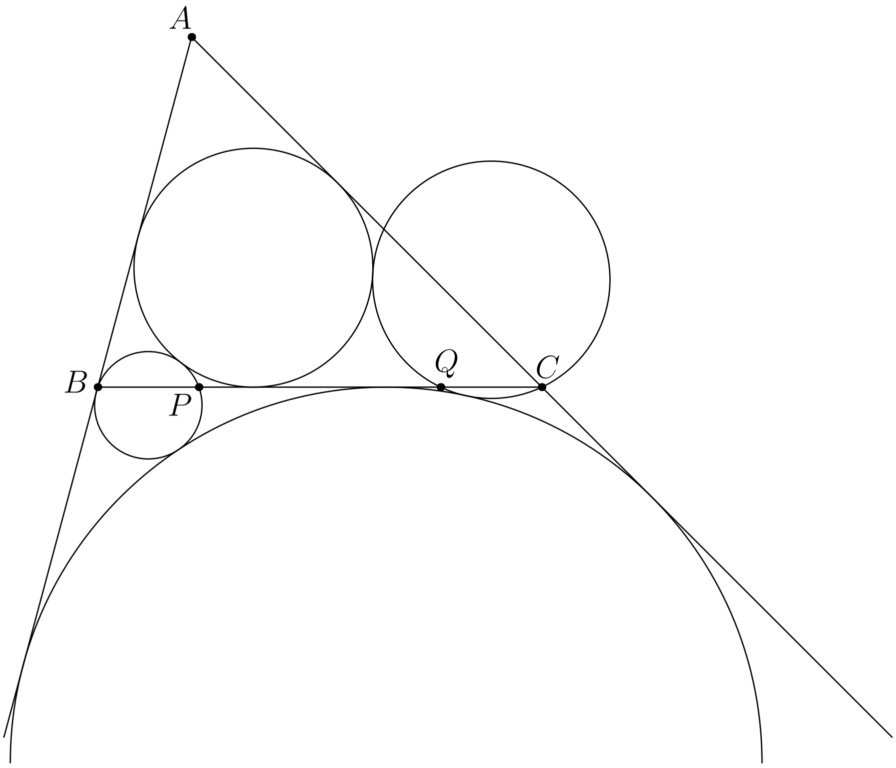
§ Problem Statement

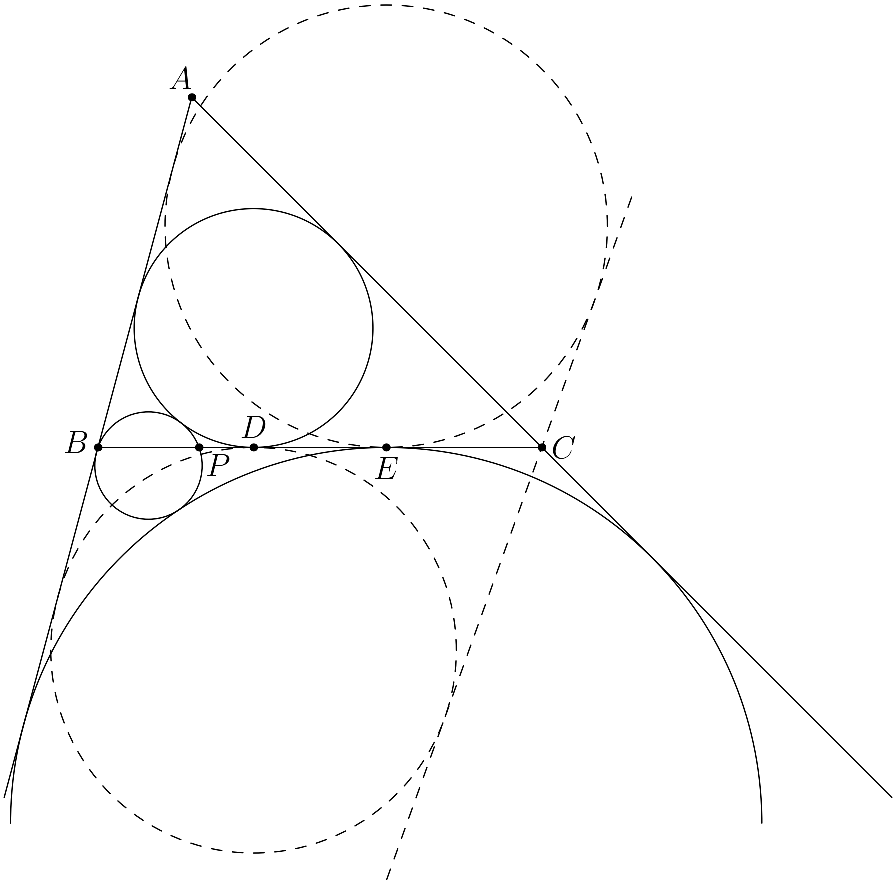
A circle through B and a circle through C are both externally tangent to the incircle and A-excircle of $\triangle ABC$. Prove $\overline{BC}$ cuts the circles into congruent chords.

§ Diagram



§ Solution

Let $\overline{BC}$ touch the incircle at D and the A-excircle at E. Since $BD = CE$, it suffices to show that $BP \cdot BC = BD \cdot BE$ by symmetry.



The inversion centered at B with radius $\sqrt{BD \cdot BE}$

- fixes lines $\overline{AB}$ and $\overline{BC}$,
- swaps D and E,
- sends the incircle to a circle Γ_1 tangent to $\overline{AB}$ and tangent to $\overline{BC}$ at E,
- sends the A-excircle to a circle Γ_2 tangent to $\overline{AB}$ and tangent to $\overline{BC}$ at D, and
- sends the circle through B tangent to the incircle and A-excircle to a common tangent ℓ of Γ_1 and Γ_2.

Now, $C \in \ell$ since ℓ must intersect $\overline{BC}$ at a point C' satisfying $BD = C'E$, so the inversion maps P to C, as desired.

§ Metadata

This problem was selected as Problem 3 of the 2023 HMIC.

- Title: Tangent to Incircle and Excircle
- Author: Holden Mui
- Subject: geometry
- Description: equal segments in configuration involving circles tangent to the incircle and A-excircle
- Keywords: incircle, excircle, tangent
- Difficulty: 20 MOHS
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