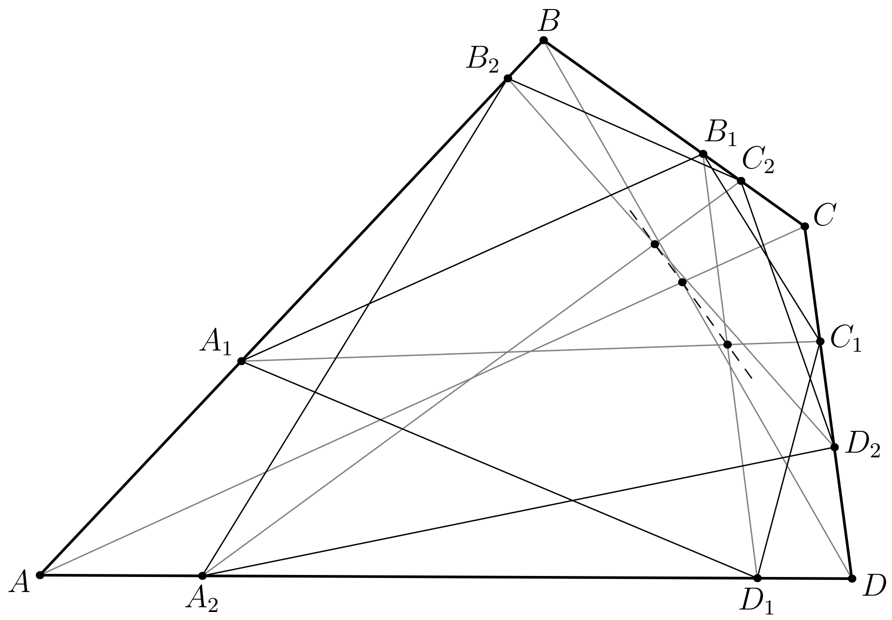
§ Problem Statement

Quadrilaterals $ABCD \sim A_1B_1C_1D_1 \sim A_2B_2C_2D_2$ lie in the plane such that

$$\overline{A_1B_2} \subset \overline{AB},\ \overline{B_1C_2} \subset \overline{BC},\ \overline{C_1D_2} \subset \overline{CD},\ \text{and } \overline{D_1A_2} \subset \overline{DA}.$$

Prove that $\overline{AC} \cap \overline{BD}$, $\overline{A_1C_1} \cap \overline{B_1D_1}$, and $\overline{A_2C_2} \cap \overline{B_2D_2}$ are collinear.

§ Diagram



§ Solution

Let P and Q be the centers of the spiral similarities from $ABCD$ to $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$, respectively, so that

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA = \theta$$

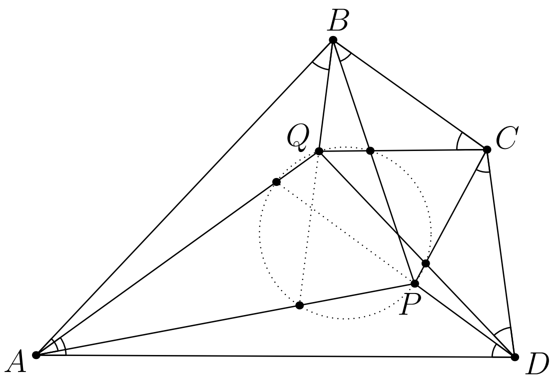
and

$$\angle QBA = \angle QCB = \angle QDC = \angle QAD = \theta';$$

note that P and Q lie inside quadrilateral $ABCD$. By the Law of Sines,

$$\begin{aligned} 1 &= \frac{\sin(\angle A - \theta)}{\sin\theta} \cdot \frac{\sin(\angle B - \theta)}{\sin\theta} \cdot \frac{\sin(\angle C - \theta)}{\sin\theta} \cdot \frac{\sin(\angle D - \theta)}{\sin\theta} \\ &= \frac{\sin(\angle A - \theta')}{\sin\theta'} \cdot \frac{\sin(\angle B - \theta')}{\sin\theta'} \cdot \frac{\sin(\angle C - \theta')}{\sin\theta'} \cdot \frac{\sin(\angle D - \theta')}{\sin\theta'} \end{aligned}$$

so monotonicity on the function $f(x) = \frac{\sin(t-x)}{\sin x}$ gives $\theta' = \theta$, as θ and θ' are both less than $\min(\angle A, \angle B, \angle C, \angle D)$.



Claim 1. *ABCD is harmonic.*

Proof. A directed angle chase gives

$$P, Q, \overline{AP} \cap \overline{BQ}, \overline{BP} \cap \overline{CQ}, \overline{CP} \cap \overline{DQ}, \overline{DP} \cap \overline{AQ}$$

concyclic, so the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ pass through an arc midpoint of PQ, proving that $ABCD$ is cyclic. Additionally,

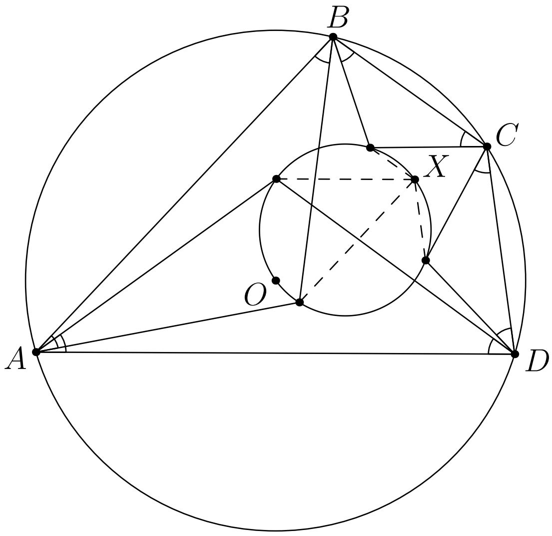
$$\triangle DAP \sim \triangle BAQ, \triangle ABP \sim \triangle CBQ, \triangle BCP \sim \triangle DCQ, \triangle CDP \sim \triangle ADQ$$

imply

$$CP = AP \cdot \frac{AB}{DA} \cdot \frac{CD}{DA} = AP \cdot \frac{BC}{AB} \cdot \frac{BC}{CD},$$

so $AB \cdot CD = BC \cdot DA$. □

Now, let O be the circumcenter of $(ABCD)$ and let $\overline{OX}$ be a diameter of (OPQ). Note that the line connecting X and $\overline{CP} \cap \overline{BQ}$ is parallel to $\overline{AB}$ since both are perpendicular to the perpendicular bisector of $\overline{AB}$, and analogous statements hold for the other three sides.



Claim 2. *$X = \overline{AC} \cap \overline{BD}$.*

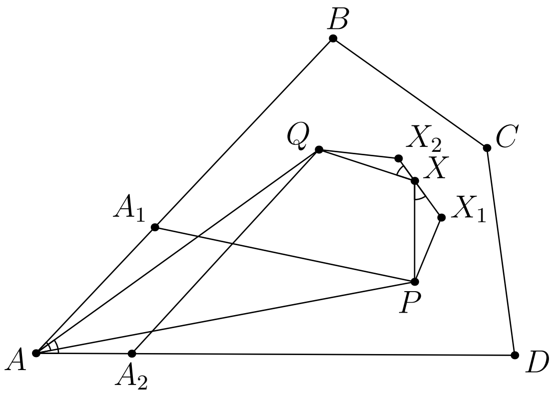
Proof. Note that

$$\begin{aligned} AB : BC : CD : DA &= \operatorname{dist}(\overline{AP} \cap \overline{BQ}, \overline{AB}) : \operatorname{dist}(\overline{BP} \cap \overline{CQ}, \overline{BC}) : \\ &\qquad \operatorname{dist}(\overline{CP} \cap \overline{DQ}, \overline{CD}) : \operatorname{dist}(\overline{DP} \cap \overline{AQ}, \overline{DA}) \\ &= \operatorname{dist}(X, \overline{AB}) : \operatorname{dist}(X, \overline{BC}) : \\ &\qquad \operatorname{dist}(X, \overline{CD}) : \operatorname{dist}(X, \overline{DA}). \end{aligned}$$

Since the intersection X' of the diagonals of a harmonic quadrilateral also satisfies

$$AB : BC : CD : DA = \operatorname{dist}(X', \overline{AB}) : \operatorname{dist}(X', \overline{BC}) : \operatorname{dist}(X', \overline{CD}) : \operatorname{dist}(X', \overline{DA})$$

the barycentric coordinates of X and X' with respect to $ABCD$ are the same. Since distinct points have distinct barycentric coordinates, $X = X'$. $\square$



Finally, note that $\angle PXQ = 180^\circ - 2\theta$ by an angle chase. Since

$$\triangle AA_1P \sim \triangle XX_1P \text{ and } \triangle AA_2Q \sim \triangle XX_2Q$$

by spiral similarity,

$$\angle XX_1X_2 = \angle X_1XP + \angle PXQ + \angle QXX_2 = \theta + (180^\circ - 2\theta) + \theta = 180^\circ,$$

implying the collinearity.

§ Variants

Variant A. Points P and Q lie inside convex quadrilateral $ABCD$ such that

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA$$

and

$$\angle QBA = \angle QCB = \angle QDC = \angle QAD.$$

Prove that $ABCD$ is cyclic.

Solution sketch. This is equivalent to the first page of the proof given above.

§ Metadata

This problem was selected as Problem 3 of the 2023 ELMO.

- Title: Similar Quadrilaterals
- Author: Holden Mui
- Subject: geometry
- Description: diagonal intersections are collinear in configuration with two inscribed quadrilaterals
- Keywords: Brocard points, collinear, quadrilateral, spiral similarity
- Difficulty: ELMO 3/6
- Collaborators: Serena An, Ankit Bisain, Vincent Huang, Michael Ren, Pitchayut Saengrungkongka, Carl Schildkraut
- Date written: October 2020
- Submission history: 2021 USEMO, 2023 ELMO