## § Problem Statement

Define a *width-one strip* to be a rectangle with width 1 or height 1. An n by n square is dissected into width-one strips such that neighboring strips share exactly one unit of perimeter. What is the minimum possible number of strips?

## § Solution

The answer is

$$\begin{cases} 1 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ \frac{1}{2}(n^2 - 2n + 7) & \text{if } n \ge 3 \text{ and } n \text{ is odd} \\ \frac{1}{2}(n^2 - 2n + 8) & \text{if } n \ge 3 \text{ and } n \text{ is even,} \end{cases}$$

with the following construction.



 $n\geq 3$  and odd

 $n \geq 3$  and even

The optimality for  $n \in \{1, 2\}$  is easy to check, so assume  $n \ge 3$ . Since the number of strips is equal to  $n^2$  minus the number of removed edges in the square, it suffices to maximize the number of removed edges.

Because no two edges can share an endpoint, at most  $\frac{1}{2}(n-3)^2$  edges can be removed within the central n-3 by n-3 grid of lattice points. Additionally, at most 4(n-2)of the outer edges can be removed since there are 4(n-2) lattice points within one unit of the boundary. Hence, the minimum number of removed edges is at least

$$n^{2} - \left( \left\lfloor \frac{1}{2}(n-3)^{2} \right\rfloor + 4(n-2) \right) = \begin{cases} \frac{1}{2}(n^{2} - 2n + 7) & \text{if } n \ge 3 \text{ and } n \text{ is odd} \\ \frac{1}{2}(n^{2} - 2n + 8) & \text{if } n \ge 3 \text{ and } n \text{ is even.} \end{cases}$$

## § Variants

**Variant A.** Define a *width-one strip* to be a rectangle with width 1 or height 1. An n by n torus is dissected into width-one strips such that neighboring strips share exactly one unit of perimeter. What is the minimum possible number of strips?

Solution sketch. The answer is  $\frac{1}{2}n^2$  when n is even and  $\frac{1}{2}(n^2 - 1)$  when n is odd; the proof is analogous to the one given above, but without the outer edge consideration.

## § Metadata

This problem was selected as Problem 1 of the 2022 USEMO.

- Title: Dissecting a Square
- Author: Holden Mui
- Subject: combinatorics
- Description: dissection of square into strips, where neighboring strips share one unit of perimeter
- Keywords: minimize, perimeter, rectangle, square, strip
- Difficulty: USEMO 1/4
- Collaborators: Kevin Cong, Ram Goel, Andrew Gu, Luke Robitaille, Carl Schildkraut, Colin Tang
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