§ Problem Statement

Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both square.

§ Solutions

Solution A

If $q = 2$, then $p - 2$ and $2(p-1)$ must both be square and cannot be $2 \pmod 3$. This forces p to be $0 \pmod 3$, so $(p, q) = \boxed{(3,2)}$ is a solution.

If $q > 2$, let $p - q = a^2$ and $pq - q = q(p-1) = b^2$. Since $q \neq b$ but $q \mid b$, $b \geq 2q$ and $p - 1 \geq 4q$. Now,

$$(b+a)(b-a) = b^2 - a^2 = pq - p = p(q-1)$$

so $b + a \geq p$ since p must divide one of the factors. Finally,

$$\sqrt{q} + 1 = \frac{\sqrt{pq} + \sqrt{p}}{\sqrt{p}} \geq \frac{\sqrt{pq - q} + \sqrt{p - q}}{\sqrt{p}} = \frac{b+a}{\sqrt{p}} \geq \frac{p}{\sqrt{p}} = \sqrt{p} \geq \sqrt{4q+1}$$

which is impossible.

Solution B

Let

$$p - q = a^2 \tag{1}$$
$$pq - q = q^2 b^2 \tag{2}$$

so that $a^2 + q = 1 + qb^2 = p$. Then

$$p^2 = (a^2+q)(1+qb^2) = (qb+a)^2 + q(ab-1)^2 \tag{3}$$
$$= (qb-a)^2 + q(ab+1)^2. \tag{4}$$

By (1) and (2), $qb^2 = p - 1 = (p - a^2)b^2$ so solving for p gives

$$p = \frac{(ab-1)(ab+1)}{b^2 - 1}.$$

Case 1. $p \mid ab - 1$. By bounding (3), $ab - 1 = 0$, which forces $a = b = 1$. This yields the solution $(p, q) = \boxed{(3,2)}$.

Case 2. $p \mid ab + 1$. By bounding (4), q must equal 1, which is not prime.

§ Comments

I wrote this problem by considering numbers that can be written in the form $x^2 + qy^2$ (fixing q) in two different ways, so the second solution above was essentially constructed backwards. Since such numbers are always composite (by using, say, quadratic number fields) but $a^2 + q = 1 + qb^2$ are two such representations for p, this forces $a = b = 1$.

§ Metadata

This problem was selected as Problem 4 of the 2022 USAMO and Problem 5 of the 2022 USAJMO.

- Title: Pairs of Primes Form Squares
- Author: Holden Mui
- Subject: number theory
- Description: p and q are primes for which $p - q$ and $pq - q$ are both square
- Keywords: primes, squares
- Difficulty: USAMO 1/4 or USAJMO 2/5
- Collaborators: Ankit Bisain, Brandon Chen, Carl Schildkraut
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- Other credits: the author of Solution A is Ankit Bisain.