§ Problem Statement

Find all $m \in \mathbb{Z}^+$ for which there exists an infinite nonconstant sequence in $\mathbb{Z}/m\mathbb{Z}$ that is both arithmetic and geometric.

§ Solutions

Solution A

m must be squareful because any three-element subsequence

$$s-d, s, s+d$$

must satisfy $s^2 \equiv (s-d)(s+d) \pmod{m}$, so $d^2 \equiv 0 \pmod{m}$. For the construction, there exists an integer n for which $m \mid n^2$ but $m \nmid n$. Then the sequence defined by

$$s_k = (n+1)^k = kn+1$$

works.

Solution B

If m is squarefree, then any arithmetic sequence with common difference d must exactly one term divisible by $\frac{n}{d}$ since $\gcd\left(d, \frac{n}{d}\right) = 1$. By the geometric series condition, all terms must be divisible by $\frac{n}{d}$, a contradiction unless $d = n$.

For the construction, let $m = kp^2$. Then the sequence

$$1, kp+1, 2kp+1, \ldots$$

works.

§ Metadata

This problem was selected as Problem 1 of the 2022 USAJMO.

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