§ Problem Statement

In terms of $n \in \mathbb{Z}^+$, find the smallest integer k for which $(0,1)^2 \setminus S$ is a union of k axis-aligned open rectangles for every set $S \subset (0,1)^2$ of size n.

§ Solution

The answer is $k = \boxed{2n+2}$.

Lower Bound

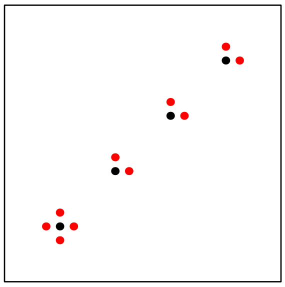
Let $\varepsilon > 0$ be sufficiently small. The lower bound is given by picking

$$S = \{(s_1, s_1), (s_2, s_2), \ldots, (s_n, s_n)\}$$

for some real numbers $0 < s_1 < s_2 < \ldots < s_n < 1$. Since no rectangle can cover more than one point in

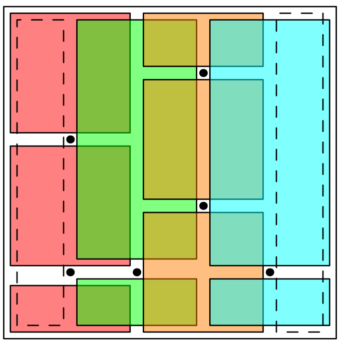
$$S' = (S + \{(\varepsilon, 0), (0, \varepsilon)\}) \cup \{(s_1 - \varepsilon, s_1), (s_1, s_1 - \varepsilon)\}$$

without covering a point in S, $k \geq |S'| = 2n + 2$.



Upper Bound

To prove that $2n + 2$ rectangles are sufficient, partition the rectangle by x-coordinate and draw rectangles as follows:



The number of rectangles used so far is $2n - a + 2$, where a is the number of pairs of points directly above/below each other.

Now, let b be the number of pairs of points directly left/right of each other, and note that all remaining uncovered points are between such pairs of points. Using b strips of small width to cover these remaining points gives a bound of $2n - a + 2 + b$ rectangles, and WLOGing $a \geq b$ finishes the problem.

§ Variants

Variant A. Let $\mathcal{T}$ denote the open unit equilateral triangle. In terms of $n \in \mathbb{Z}^+$, find the smallest integer k for which the $\mathcal{T} \setminus S$ is a union of k axis-aligned equilateral triangles for every set $S \subset (0,1)^2$ of size n.

Solution sketch. I do not know what the answer to this variant is.

Variant B. In terms of $n \in \mathbb{Z}^+$, find the smallest integer k for which $\mathbb{R}^2 \setminus S$ is a union of k open convex regions for every set $S \subset (0,1)^2$ of size n.

Solution sketch. I do not know what the answer to this variant is.

Variant C. Fix $n \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$, and define an *open prism* to be a product of d open intervals. In terms of n and d, find the smallest integer k for which $(0,1)^d \setminus S$ is a union of k open prisms for every set $S \subset (0,1)^d$ of size n.

Solution sketch. The answer is probably $d(n-1)$, though I do not know how to prove it.

§ Comments

I came up with this problem during one of my topology lectures, where we were discussing countable bases for the standard topology over $\mathbb{R}^2$. Since removing a finite number of points from $\mathbb{R}^2$ leaves an open set, it was natural to ask what the minimal number of rectangles needed to cover the set might be.

§ Metadata

This problem was selected as Problem 1 of the 2022 TSTST.

- Title: Covering $(0,1)^2$
- Author: Holden Mui
- Subject: combinatorics
- Description: minimum number of axis-aligned open rectangles needed to cover $(0,1)^2$ with k points removed
- Keywords: cover, open, rectangle, union
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