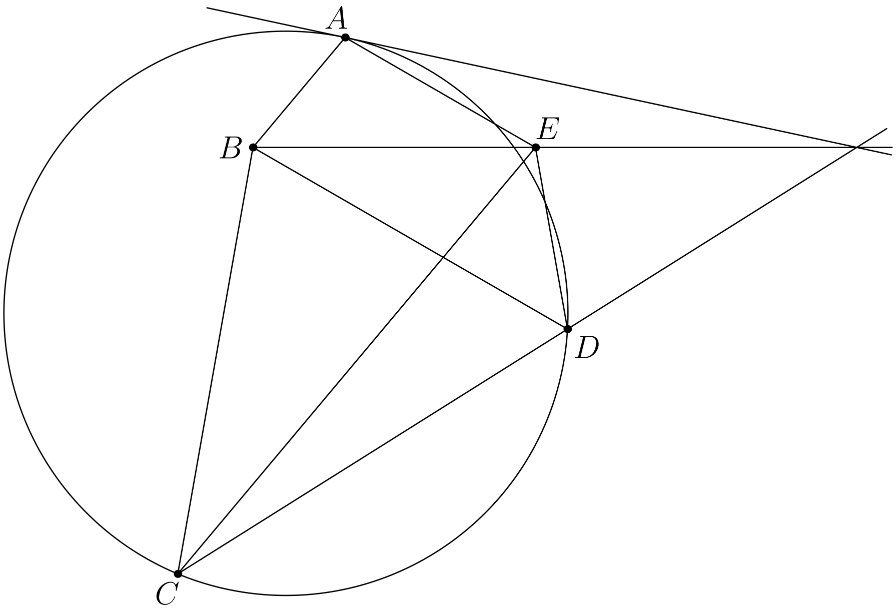
§ Problem Statement

Convex pentagon $ABCDE$ satisfies $ABE \sim BEC \sim EDB$. Prove $\overline{BE}$, $\overline{CD}$, and the tangent to (ACD) at A concur.

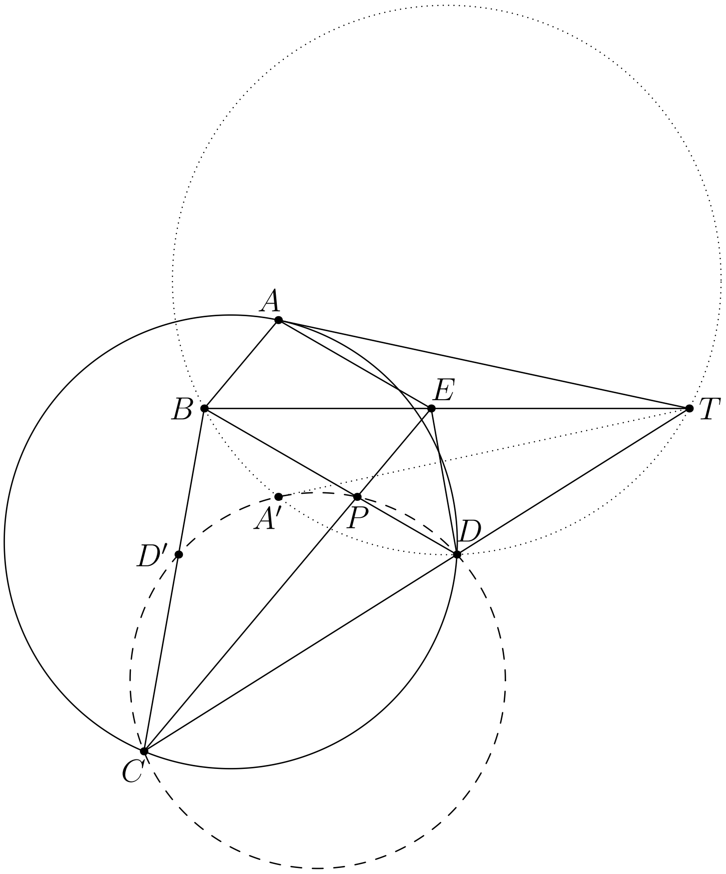
§ Diagram



§ Solutions

Solution A

Let $P = \overline{BC} \cap \overline{CE}$, $T = \overline{BE} \cap \overline{CD}$, A' be the reflection of A over $\overline{BE}$, and D' be the reflection of D over the perpendicular bisector of $\overline{BE}$. Note that $D' \in \overline{BC}$ and $\overline{BE} \parallel \overline{A'P} \parallel \overline{DD'}$.



Since

$$\measuredangle D'CP = \measuredangle EBD = \measuredangle D'DP,$$

$D' \in (CDP)$, so $A' \in (CDP)$. Since

$$\measuredangle A'BT = \measuredangle BEC = \measuredangle A'PC = \measuredangle A'DT,$$

$A' \in (BDT)$. Since

$$\measuredangle DA'T = \measuredangle DBT = \measuredangle BCE = \measuredangle A'CD,$$

$A'T$ is tangent to $(A'CD)$. Finally,

$$CT \cdot DT = A'T^2 = AT^2$$

so $\overline{AT}$ is tangent to (ACD).

Solution B

Let T be the intersection of $\overline{CD}$ and the tangent to (ACD) at A. Since $\measuredangle ABD = \measuredangle AEC$ and

$$\frac{AB}{BD} = \frac{AB \cdot AE}{BE^2} = \frac{AE}{EC},$$

$\triangle ABD \sim \triangle AEC$. Then

$$\frac{[BCE]}{[BDE]} = \left(\frac{AE}{AB}\right)^2 = \left(\frac{AC}{AD}\right)^2 = \frac{TC}{TA} \cdot \frac{TA}{TD} = \frac{TC}{TD},$$

so $T \in \overline{BE}$ as desired.

Solution C

Observe that $\overline{BC}$ and $\overline{DE}$ are tangent to (ABE). After $\sqrt{be}$-inversion at A, the problem becomes:

> Let ABE be a triangle. Construct C' so that $(AC'E)$ is tangent to $\overline{BE}$ and $(AC'B)$ is tangent to $\overline{AE}$. Construct D' so that $(AD'E)$ is tangent to $\overline{AB}$ and $(AD'B)$ is tangent to $\overline{BE}$. Let $(AC'D')$ meet (ABE) again at T'. Show that $\overline{AT'} \parallel \overline{C'D'}$.

The tangency conditions imply that C' and D' are the Brocard points of $\triangle ABE$, so are equidistant from ABE's circumcenter. Therefore $\overline{AT'} \parallel \overline{C'D'}$, as desired.

Solution D

Let $T = \overline{BE} \cap \overline{CD}$, and note that C and D are isogonal conjugates in $\triangle ABE$. Let $\infty_B \in \overline{AE}$, $\infty_E \in \overline{AB}$, and $\infty \in \overline{CD}$ be points at infinity. By Desargues' involution theorem on quadrangle $BE\infty_B\infty_E$ and $\overline{CD}$, there exists an involution swapping $\{T, \infty\}$, $\{C, D\}$, and $\{AB \cap CD, AE \cap CD\}$. Projecting at A shows that the latter two pairs identify this involution as isogonal conjugation in $\angle BAE$. Therefore $\overline{AT}$ and $\overline{A\infty}$ are isogonal in $\angle BAE$, so $\overline{AT}$ is tangent to (ACD).

§ Comments

I wrote this problem with the intention to make a geometry problem with as few named points as possible. Originally all the triangles are similarly oriented, but I realized the problem was also true if one of the triangles was oppositely oriented.

§ Metadata

This problem was selected as Problem 4 of the 2022 ELMO. It appeared as G2 on the 2021 ELMO Shortlist and as G2 on the 2022 ELMO Shortlist.

- Title: Convex Pentagon
- Author: Holden Mui
- Subject: geometry
- Description: prove three lines concur on diagram involving a convex pentagon and three similar triangles
- Keywords: concurrent, pentagon, similar triangles, tangent
- Difficulty: easy ELMO 2/5
- Collaborators: Carl Schildkraut, Colin Tang
- Date written: November 2020
- Submission history: 2021 ELMO, 2022 ELMO
- Other credits: the author of Solution B is Eric Shen, the author of Solution C is Andrew Gu, and the author of Solution D is Noah Walsh.