§ Problem Statement

In an n by n square of n^2 real numbers, maximize the number of numbers greater than the average of its row but less than the average of its column.

§ Solution

The answer is $\boxed{(n-1)^2}$, achieved with the construction below.

$$\begin{matrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{matrix}$$

To prove optimality, perturb each entry by an infinitesimally small amount so that all entries are distinct, and call a cell *unbalanced* if it is the smallest in its row and the largest number in its column. There is at most one unbalanced cell, since the existence of two unbalanced cells gives an impossible chain of inequalities.

Hence, the set of cells that are the smallest in their row must share at most one cell with the set of entries that are the largest in their column, so at least $2n-1$ cells cannot satisfy the problem condition, giving a bound of at most

$$n^2 - (2n-1) = (n-1)^2.$$

§ Metadata

This problem was selected as Problem 1 of the 2021 USEMO.

- Title: Balanced Cells
- Author: Holden Mui
- Subject: combinatorics
- Description: in a square grid, maximize the number of numbers greater than its row average but less than its column average
- Keywords: average, column, row, square grid
- Difficulty: USEMO 1/4
- Collaborators: Carl Schildkraut
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