§ Problem Statement

Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0,0)$ and ending at a point on the line $x+y=n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths.

§ Solution

Induct on n; the base case $n = 1$ is obvious. For the inductive step, there are three cases.

- All mine-avoiding paths pass through $(1, 0)$. Then there must be a element of M with zero x-coordinate, or else the y-axis would be a mine-avoiding path that avoids $(1, 0)$. Since this element of M doesn't affect any paths passing through $(1, 0)$, there are at least
$$2^{(n-1)-(|M|-1)} = 2^{n-|M|}$$
mine-avoiding paths passing through $(1, 0)$.
- All mine-avoiding paths pass through $(0, 1)$. This is analogous to the previous case.
- There are mine-avoiding paths passing through $(1, 0)$ and $(0, 1)$. Then there are at least $2^{n-1-|M|}$ mine-avoiding paths passing through $(1, 0)$, and similarly for $(0, 1)$, which gives a total of at least
$$2^{(n-1)-|M|} + 2^{(n-1)-|M|} = 2^{n-|M|}$$
mine-avoiding paths.

§ Comments

Ankit sent me a version of this problem involving a square grid instead of a triangular grid. I simply noted that the problem could be strengthened to the triangular grid version.

§ Metadata

This problem was selected as Problem 7 of the 2021 TSTST.

- Title: Counting Paths
- Authors: Ankit Bisain, Holden Mui
- Subject: combinatorics
- Description: mine-avoiding lattice paths from $(0, 0)$ to $x + y = n$
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