§ Problem Statement

Let a and b be positive integers such that there are infinitely many pairs of positive integers (m, n) such that m^2+an+b and n^2+am+b are both square. Prove that $a \mid 2b$.

§ Solution

Write the system as

$$\begin{aligned} m^2 + an + b &= (m+r)^2 \\ n^2 + am + b &= (n+s)^2 \end{aligned}$$

for some positive integers r and s.

Claim 1. *There are only finitely many pairs (r, s) for which there are solutions to the system.*

Proof. Summing the two equations and simplifying gives

$$(a-2r)m + (a-2s)n = r^2 + s^2 - 2b.$$

If $a - 2r > 0$, then solving for $\frac{m}{n}$ in two ways after noting $an + b \geq 2m + 1$ gives

$$\frac{2s-a}{a-2r} + \frac{r^2+s^2-2b}{n(a-2r)} = \frac{m}{n} \leq \frac{a}{2} + \frac{b-1}{2n}$$

so for each choice of $0 < r < \frac{a}{2}$, there are only finitely many choices for s. A similar result holds when $a - 2s > 0$.

If both $a - 2r$ and $a - 2s$ are less than or equal to zero, then the left side of the first equation cannot be positive, implying that $r^2 + s^2 \leq 2b$ and thus there are only finitely many pairs (r, s) that yield solutions. □

Simplifying the system gives

$$\begin{aligned} an &= 2rm + r^2 - b \\ 2sn &= am + b - s^2 \end{aligned}$$

Since the system is linear, for there to be infinitely many solutions (m, n) the system must be dependent. This gives

$$\frac{a}{2s} = \frac{2r}{a} = \frac{r^2 - b}{b - s^2}$$

so $a = 2\sqrt{rs}$ and $b = \frac{s^2\sqrt{r} + r^2\sqrt{s}}{\sqrt{r}+\sqrt{s}}$. Since rs must be square, we can reparametrize as $r = kx^2$, $s = ky^2$, and $\gcd(x, y) = 1$. This gives

$$\begin{aligned} a &= 2kxy \\ b &= k^2xy(x^2 - xy + y^2). \end{aligned}$$

Thus, $a \mid 2b$, as desired.

§ Variants

Variant A. Find all positive integers k such that the number of pairs of distinct positive integers (a, b) such that $a^2 + kb$ and $b^2 + ka$ are both squares is

(a) at least one.

(b) infinite.

Solution. The answer to part (b) is $\boxed{\text{no such } k \text{ exists}}$. It suffices to show there are finitely many positive integer solutions to the system

$$a^2 + kb = (a + m)^2$$
$$b^2 + ka = (b + n)^2$$

with $m \leq n$. Summing the two equations gives

$$(k - 2m)a + (k - 2n)b = m^2 + n^2$$

and rearranging gives

$$\frac{a}{b} = \frac{2n - k}{k - 2m} + \frac{m^2 + n^2}{b(k - 2m)}.$$

Since $k - 2m > 0$ from the summed equations and $kb \geq 2a + 1$ because $m \geq 1$, chaining inequalities gives

$$\frac{k}{2} > \frac{a}{b} > \frac{2n - k}{k - 2m},$$

implying that for each of the finitely many choices for m, there are only finitely choices for n. Since each choice of m and n gives a system of linearly independent linear equations in a and b, it follows that there are only finitely many pairs (a, b) that satisfy the condition for every k.

§ Metadata

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