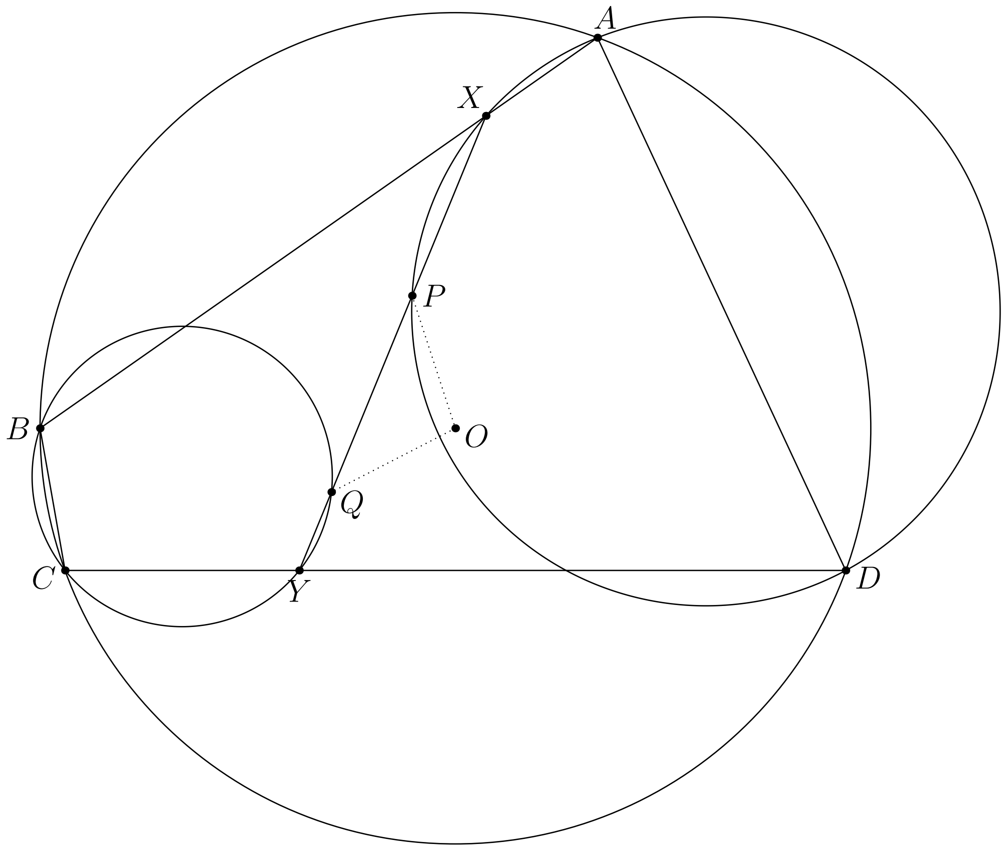
§ Problem Statement

Points X and Y lie on sides $\overline{AB}$ and $\overline{CD}$ of cyclic quadrilateral $ABCD$ with center O. If (ADX) and (BCY) meet $\overline{XY}$ again at P and Q, prove $OP = OQ$.

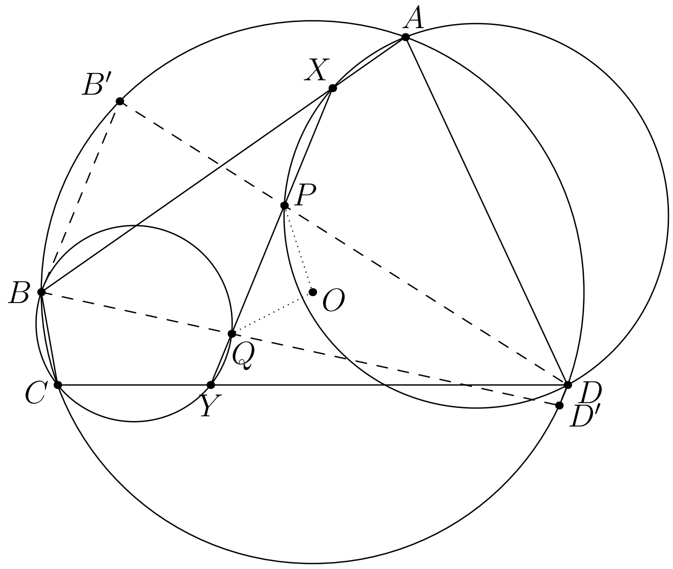
§ Diagram



§ Solutions

Solution A

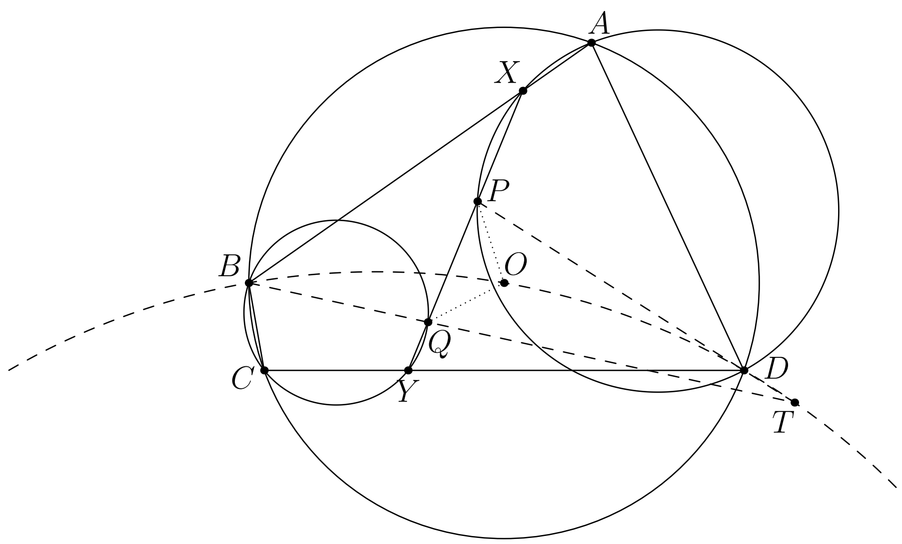
Let $\overline{BQ} \cap (ABCD) = D'$ and $\overline{DP} \cap (ABCD) = B'$.



Then $\overline{BB'} \parallel \overline{PX}$ and $\overline{DD'} \parallel \overline{QY}$ by Reim's theorem, so the result follows by symmetry on isosceles trapezoid $BB'DD'$.

Solution B

Let $T = \overline{BQ} \cap \overline{DP}$.



Note that PQT is isosceles because

- $\measuredangle PQT = \measuredangle YQB = \measuredangle BCD$ and
- $\measuredangle TPQ = \measuredangle XPD = \measuredangle BAD = \measuredangle BCD$.

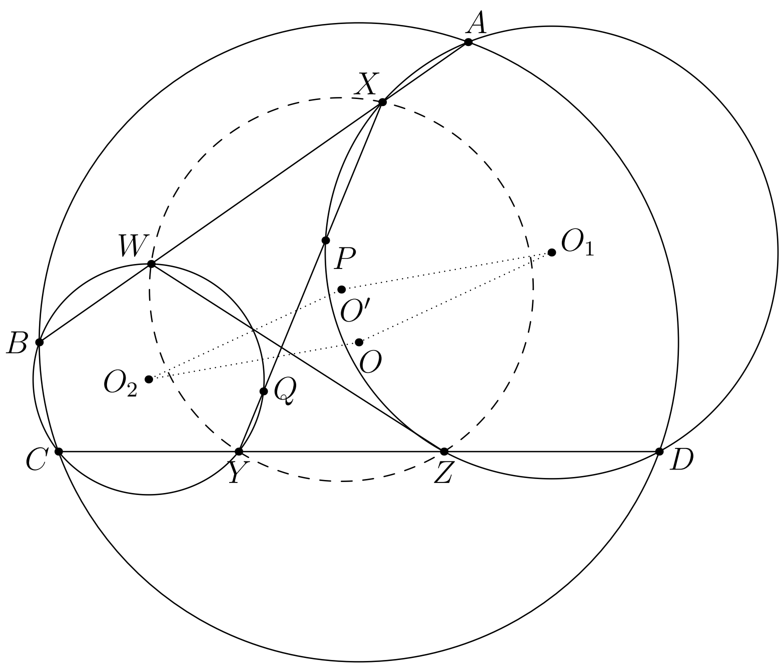
Then $(BODT)$ is cyclic because

$$\measuredangle BOD = 2\measuredangle BCD = \measuredangle PQT + \measuredangle TPQ = \measuredangle BTD.$$

Since $BO = OD$, $\overline{TO}$ is an angle bisector of $\measuredangle BTD$. Since $\triangle PQT$ is isosceles, $\overline{TO} \perp \overline{PQ}$, so $OP = OQ$.

Solution C

Let (BCY) meet $\overline{AB}$ again at W and let (ADX) meet $\overline{CD}$ again at Z. Additionally, let O_1 be the center of (ADX) and O_2 be the center of (BCY).



Note that $(WXYZ)$ is cyclic since

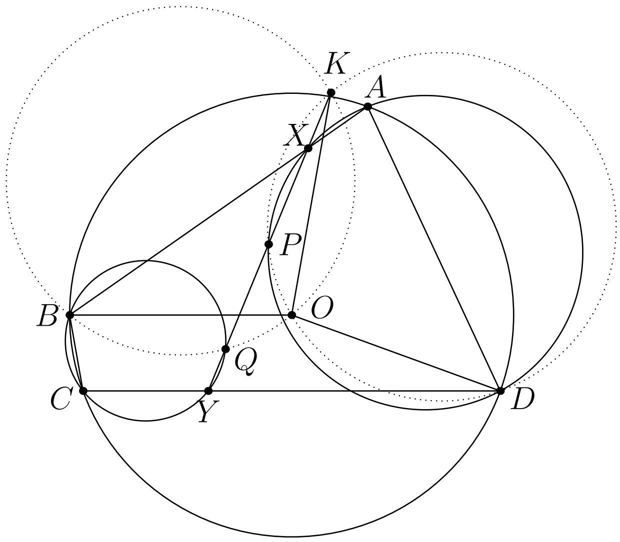
$$\measuredangle XWY + \measuredangle YZX = \measuredangle YWB + \measuredangle XZD = \measuredangle YCB + \measuredangle XAD = 0^\circ,$$

so let O' be the center of $(WXYZ)$. Since $\overline{AD} \parallel \overline{WY}$ and $\overline{BC} \parallel \overline{XZ}$ by Reim's theorem, $OO_1O'O_2$ is a parallelogram.

To finish the problem, note that projecting O_1, O_2, and O' onto $\overline{XY}$ gives the midpoints of $\overline{PX}$, $\overline{QY}$, and $\overline{XY}$. Since $OO_1O'O_2$ is a parallelogram, projecting O onto $\overline{XY}$ must give the midpoint of $\overline{PQ}$, so $OP = OQ$.

Solution D

Let the angle bisector of $\measuredangle BOD$ meet $\overline{XY}$ at K.



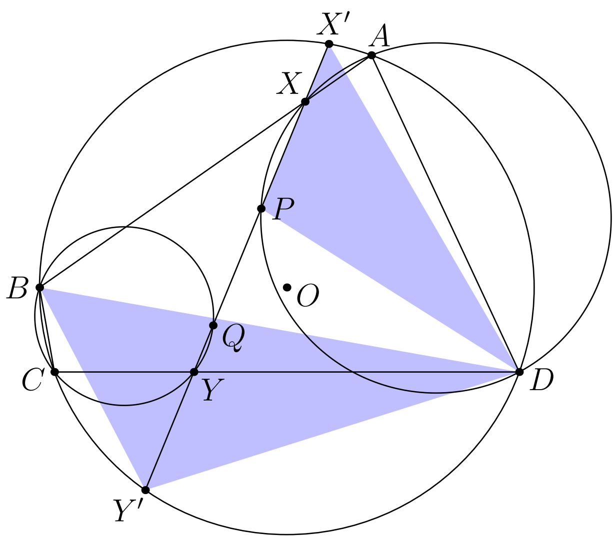
Then $(BQOK)$ is cyclic because $\measuredangle KOD = \measuredangle BAD = \measuredangle KPD$, and $(DOPK)$ is cyclic similarly. By symmetry over KO, these circles have the same radius r, so

$$OP = 2r \sin \angle OKP = 2r \sin \angle OKQ = OQ$$

by the Law of Sines.

Solution E

Let $\overline{XY}$ meet $(ABCD)$ at X' and Y'.



Since $\measuredangle Y'BD = \measuredangle PX'D$ and $\measuredangle BY'D = \measuredangle BAD = \measuredangle X'PD$, $BY'D \sim X'PD$, so

$$PX' = BY' \cdot \frac{DX'}{BD}.$$

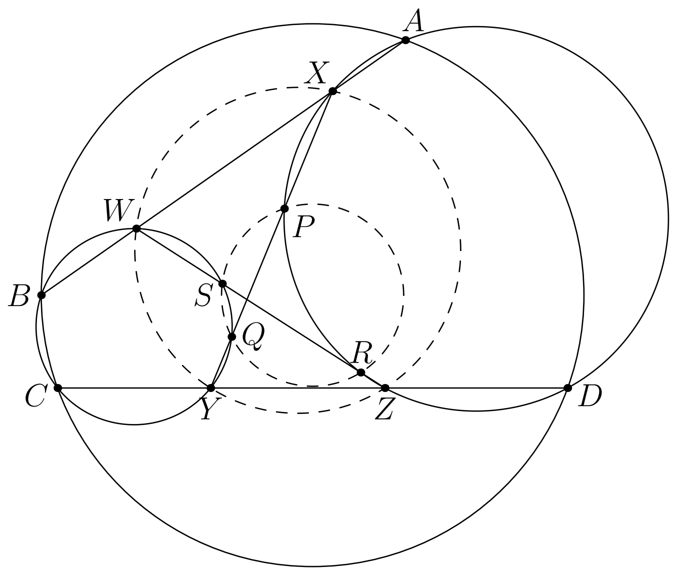
Similarly, $BX'D = BQY'$, so

$$QY' = DX' \cdot \frac{BY'}{BD}.$$

Thus $PX' = QY'$, which gives $OP = OQ$.

Solution F

Without loss of generality, assume $\overline{AD} \nparallel \overline{BC}$, as this case holds by continuity. Let (BCY) meet $\overline{AB}$ again at W, let (ADX) meet $\overline{CD}$ again at Z, and let $\overline{WZ}$ meet (ADX) and (BCY) again at R and S.



Note that $(WXYZ)$ is cyclic since

$$\measuredangle XWY + \measuredangle YZX = \measuredangle YWB + \measuredangle XZD = \measuredangle YCB + \measuredangle XAD = 0^\circ$$

and $(PQRS)$ is cyclic since

$$\measuredangle PQS = \measuredangle YQS = \measuredangle YWS = \measuredangle PXZ = \measuredangle PRZ = \measuredangle SRP.$$

Additionally, $\overline{AD} \parallel \overline{PR}$ since

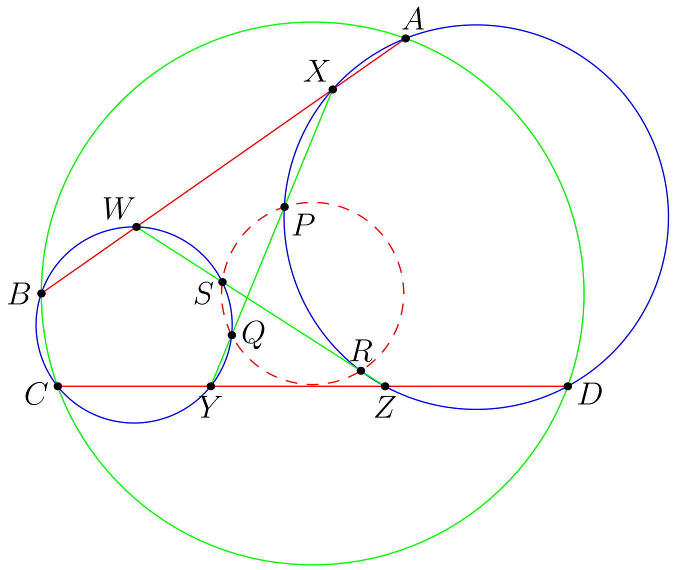
$$\measuredangle DAX + \measuredangle AXP + \measuredangle XPR = \measuredangle YWX + \measuredangle WXY + \measuredangle XYW = 0^\circ,$$

and $\overline{BC} \parallel \overline{SQ}$ similarly.

Lastly, $(ABCD)$ and $(PQRS)$ are concentric; if not, using the radical axis theorem twice shows that their radical axis must be parallel to both $\overline{AD}$ and $\overline{BC}$, contradiction.

Solution G

Let (BCY) meet $\overline{AB}$ again at W, let (ADX) meet $\overline{CD}$ again at Z, and let $\overline{WZ}$ meet (ADX) and (BCY) again at R and S.



The quartics $(ADXZ) \cup (BCWY)$ and $\overline{XY} \cup \overline{WZ} \cup (ABCD)$ meet at the 16 points

$$A, B, C, D, W, X, Y, Z, P, Q, R, S, I, I, J, J,$$

where I and J are the circular points at infinity. Since $\overline{AB} \cup \overline{CD} \cup (PQR)$ contains the 13 points

$$A, B, C, D, P, Q, R, W, X, Y, Z, I, J,$$

it must contain S, I, and J as well, by quartic Cayley-Bacharach. Thus, $(PQRS)$ is cyclic and intersects $(ABCD)$ at I, I, J, and J, implying that the two circles are concentric, as desired.

§ Metadata

This problem was selected as Problem 1 of the 2021 TSTST.

- Title: Equidistant from Circumcenter
- Author: Holden Mui
- Subject: geometry
- Description: in cyclic quadrilateral, prove two points are equidistant from circumcenter
- Keywords: circumcircle, cyclic quadrilateral, equidistant
- Difficulty: TSTST 1/4/7
- Collaborators: Ankit Bisain, Carl Schildkraut, Colin Tang
- Date written: April 2021
- Submission history: 2021 TSTST
- Other credits: the author of Solution A is Ankit Bisain, and the contestants found Solutions B, C, D, and E.