§ Problem Statement

Evan has a convex n-gon in the plane for a given positive integer $n \geq 3$ and wishes to construct the centroid of its vertices. He has no standard ruler or compass, but he does have a device with which he can dissect the segment between two given points into m equal parts. For what m can Evan necessarily accomplish his task?

§ Solution

The answer is all m for which $\operatorname{rad}(2n) \mid m$. Define an k-dissector to be a device that can dissect the segment between two given points into k equal parts, and let the vertices of the n-gon be represented by the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. The goal is to construct $\frac{1}{n}\left(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \ldots + \boldsymbol{v}_n\right)$.

Necessity

Since a applications of a m-dissector produces a point of the form

$$\frac{c_1\boldsymbol{v_1} + \ldots + c_n\boldsymbol{v_n}}{m^a},$$

$n \mid m^i$ for some sufficiently large a, so $\operatorname{rad}(n) \mid m$. To show that $2 \mid m$, note that the only points achievable when $2 \nmid m$ have exactly one odd c_i by interpreting coordinates modulo 2. Therefore $\operatorname{rad}(2n) \mid m$.

Sufficiency

Clearly a $\operatorname{rad}(2n)$-dissector can construct what both a 2-dissector and an n-dissector can construct, so assume we have a 2-dissector and an n-dissector instead.

Let 2^a be the smallest power of 2 greater than or equal to n. Define the 2^a vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{2^a}$ as follows, where $\{x\}$ denotes the fractional part of x:

$$\boldsymbol{u}_i = \begin{cases} \left\{\frac{2^a}{n} \cdot k\right\} \cdot \boldsymbol{v}_k + \left(1 - \left\{\frac{2^a}{n} \cdot k\right\}\right) \cdot \boldsymbol{v}_{k+1} & \text{if } i - 1 < \frac{2^a}{n} \cdot k < i \\ \boldsymbol{v}_k & \text{if } \frac{2^a}{n} \cdot k \le i - 1 \text{ and } i \le \frac{2^a}{n} \cdot (k+1) \end{cases}$$

Each of these vectors can be constructed using the n-dissector. Since these 2^a vectors all sum to $\frac{2^a}{n} \cdot (\boldsymbol{v}_1 + \boldsymbol{v}_2 + \ldots + \boldsymbol{v}_n)$, the centroid of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{2^a}$ is the same of the centroid of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. To finish, just use the 2-dissector to construct the centroid of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{2^a}$.

§ Metadata

This problem was selected as Problem 2 of the 2019 ELMO.

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- Subject: algebra, combinatorics, number theory
- Description: constructing the centroid of a set of points using a special ruler
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