Physics Napkin

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0 Introduction

In the middle of my senior year of high school, I started thinking about the suboptimal way in which high school physics is generally taught. Sure, I score well on the tests because I can regurgitate formulas and apply them, but I've always felt that by learning physics in this fashion, I was somehow cheating myself out of a more "true," meaningful physics experience. As an avid lover of mathematical rigor and problem solving, this bothered me. Given the current way our physics curriculum is taught, it is nearly impossible for students, myself included, to be able to solve a problem that they've never seen before.

One might object to this philosophy and ask, "why do we need to know how to solve physics problems we haven't seen before, if they won't ever appear on a test? All the test problems are just variations on problems from the homework." Don't get me wrong; this is entirely correct! This philosophy definitely works in a school setting.

However, this philosophy fails miserably in practically every setting that is not a school setting. Life is not school. One cannot complain "but the teacher never taught me this in class!" when life throws an unanticipated problem; this is why acquiring strong problem-solving skills and developing resiliency is so valuable. While an abstract physics problem may seem only tangentially related to making an important decision in the workplace, the inherent processes used to deal with each one are really just fundamentally isomorphic.

I think the correct answer to students asking the question "why will I need to know this in life" when faced with learning, say, Newton's second law, is not because they will actually need it in the career, but rather because the problem solving skills and resiliency a student develops from attempting to cope with such an abstract concept are precisely the same skills one needs to know to cope with a "real-life" problem.

I am of the opinion that the development of problem solving skills in high school physics could be taken further. In its current state, we are usually taught a set of formulas, which will then be used to solve the problems on the test; in other words, we only need to know the formulas and not the logical argument for their existence. While solving such "formula-application" problems may be mildly interesting to some, there are much more intriguing questions to consider that are never answered in class. Why is defining torque as a cross product a useful construct to consider? Why are the orbits of planets elliptical? (3blue1brown has a great video about this!)

A deeper understanding of the methodologies used to develop the tools that we use to solve high school physics problems will enable one to understand how one could develop one's own arsenal of tools in any discipline as well.

With this philosophy in mind, I came to the conclusion that to really get a solid understanding of the foundations of physics, I needed to start at the beginning – the very beginning. I would start from the bare minimum number of assumptions about the nature of reality – which turn out to just be Newton's three laws (and a few other assumptions) – and see how much of the physical world I could model just by using theoretical arguments on the "axioms," much in the same way one might deduce all of Euclidean geometry from Euclid's five postulates. I was adamant about doubting the truth of all statements until I was able to prove it; one might call it the opposite of indoctrination. For example, I would not accept on faith that the kinetic energy of an object was $\frac{1}{2}mv^2$; I had to prove it, from the definitions, whatever they might be.

In this way, many of my misconceptions about classical mechanics were corrected.

This long exercise also forced me to consider every small detail about physics that may be overlooked in a general physics course. For example,

- Everyone who has taken physics knows that $\mathbf{F} = m\mathbf{a}$, where the force is applied towards the object's center. Is $\mathbf{F} = m\mathbf{a}$ still true if one applies a force at an angle that does not point towards or away from the center of an object?
- Is $\mathbf{F} = m\mathbf{a}$ still true when the mass of an object is changing? A good example of this is a rocket that is losing fuel.
- Newton's first law states that an object in motion stays in motion and an object at rest stays at rest unless a force is acted upon it. Newton's second law states that $\mathbf{F} = m\mathbf{a}$. Does Newton's second law follow from Newton's first law by setting $\mathbf{F} = 0$? If so, what would Newton come up with a "law" that can be derived from another law?
- Which of the rotational motion formulae are derivable from Newton's laws, and which ones must be accepted as fact, without proof? More generally, which formulas on the AP Physics formula sheet are derivable through calculation, and which ones are just laws extrapolated from observation?
- Suppose a bullet is shot upwards and gets lodged in a block of wood near its edge, sending the block and the bullet inside it spinning. Before the collision, no objects were spinning, but after the collision, both objects spin. Does this violate conservation of angular momentum?

The answers to each of the above questions should be obvious to anyone with a solid conceptual understanding of physics, but before I started on my quest, the answers to each of these questions were not obvious at all. The slow process of questioning what should be true and figuring out why things should be true is what led me to discover this deeper understanding for myself.

I have documented my journey, definition by definition and proof by proof, that I've taken to rigorously build physics from the ground up. The result is this exposition. It represents the way I wish I was taught physics in high school.

§0.1 Acknowledgments

The style in the document is modeled after Evan Chen's *Napkin*. I started writing this exposition as part of Naperville North's STEM Capstone program. Practice problems are primarily sourced from the American Association of Physics Teachers' annual $\mathbf{F} = m\mathbf{a}$ exam.

I

Classical Mechanics

Part I: Contents

1 Kinematics

Kinematics is the geometry of motion. All motion is assumed to take place in the three-dimensional vector space $V = \mathbb{R}^3$ equipped with the Euclidean norm. All objects in kinematics are modeled as **point particles**, which are idealized particles with no volume, or **rigid bodies**, which are collections of point particles for which the particles do not move relative to each other.

§1.1 Time

Prototypical example for this section: 1 January 1970 00:00:00 UTC.

Time is what a clock reads. It is a fundamental scalar quantity measured in **seconds**, abbreviated “s”.

> **Remark 1.1.1** — A second is approximately $\frac{1}{24 \cdot 60 \cdot 60} = \frac{1}{86400}$ of a solar day.

The positions, velocities, and accelerations of point particles in kinematics evolve over time and are represented as functions of time.

§1.2 Position

Prototypical example for this section: what a GPS reads.

Every point in space can be represented as a position vector drawn from the origin of the vector space to the point. The **distance** between two points represented by position vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\|\mathbf{x}_2 - \mathbf{x}_1\| .$$

Distance is a fundamental scalar quantity measured in **meters**, abbreviated “m”.

> **Remark 1.2.1** — A meter is approximately $\frac{1}{10^7}$ of the distance between Earth's North Pole and its equator along a meridian.

The **trajectory** of a point particle can be modeled as a vector-valued function $\mathbf{x}(t) : \mathbb{R} \to V$, where t is a parameter representing time. It defines the curve traced by the particle; that is, $\mathbf{x}(t)$ is the position of the point particle at time t. Trajectories are assumed to be continuous functions.

Abuse of Notation 1.2.2. The parameter t in functions is often omitted. For example, $\mathbf{x}$ will be understood to be the position $\mathbf{x}(t)$ of a particle as a function of time.

The **displacement** of a point particle with trajectory $\mathbf{x}$ between times t_1 and t_2 is the vector

$$\mathbf{x}(t_2) - \mathbf{x}(t_1).$$

Displacement is measured in meters.

§1.3 Velocity

Prototypical example for this section: what a speedometer and a compass collectively read.

The **average velocity** of a point particle over a period of time is its displacement divided by the duration of the time interval; that is, the average velocity between times t_1 and t_2 is the vector

$$\frac{\mathbf{x}(t_2) - \mathbf{x}(t_1)}{t_2 - t_1}.$$

The **instantaneous velocity** of a point particle at time t, abbreviated as just **velocity**, is the limit of the average velocity of the point particle between times t and t' as t' approaches t. In other words, velocity is the derivative of position:

$$\mathbf{v}(t) = \lim_{t' \to t} \frac{\mathbf{x}(t') - \mathbf{x}(t)}{t' - t} = \frac{d\mathbf{x}}{dt}.$$

It is a vector-valued function that describes the magnitude as well as the direction of motion of the particle. Average velocity and velocity are both measured in **meters per second**, abbreviated "$\frac{\mathrm{m}}{\mathrm{s}}$".

Proposition 1.3.1 (Integral of velocity is displacement)
Displacement is the integral of velocity; that is,

$$\mathbf{x}(t_2) - \mathbf{x}(t_1) = \int_{t_1}^{t_2} \mathbf{v}(t)dt.$$

Proof. Since $\mathbf{v} = \frac{d\mathbf{x}}{dt}$, this follows from the Fundamental Theorem of Calculus. □

§1.4 Speed

Prototypical example for this section: what a speedometer reads.

The **speed** of a point particle is a scalar quantity representing the magnitude of its velocity. The function $v : \mathbf{x} \to \mathbb{R}_{\geq 0}$ representing the speed of a particle with velocity $\mathbf{v}$ is

$$v(t) = \|\mathbf{v}(t)\|.$$

Like average velocity and velocity, the units of speed are meters per second.

The **distance** traveled by a point particle with trajectory $\mathbf{x}(t)$ between times t_1 and t_2 is the arc length of its trajectory, which is

$$\int_{t_1}^{t_2} \|\mathbf{v}(t)\| \, dt = \int_{t_1}^{t_2} v(t)dt.$$

Distance is measured in meters.

The **average speed** of a point particle over a period of time is its distance traveled divided by the duration of the time interval; that is, the average speed between times t_1 and t_2 is

$$\frac{\int_{t_1}^{t_2} v(t)dt}{t_2 - t_1}.$$

Average speed is measured in meters per second.

> **Exercise 1.4.1** (distance equals rate times time)**.** Explain why for a point particle moving at a constant speed, distance is the product of speed and time.

§1.5 Acceleration

Prototypical example for this section: pressing the gas pedal, applying the brakes, or turning the steering wheel while driving.

The **average acceleration** of a point particle over a period of time is its change in velocity divided by the duration of the time interval; that is, the average velocity between times t_1 and t_2 is the vector

$$\frac{\mathbf{v}(t_2) - \mathbf{v}(t_1)}{t_2 - t_1}.$$

The **instantaneous acceleration** of a point particle at time t, abbreviated as just **acceleration**, is the limit of the average acceleration of the point particle between times t and t' as t' approaches t. Acceleration is the derivative of velocity and the second derivative of position:

$$\mathbf{a}(t) = \lim_{t' \to t} \frac{\mathbf{v}(t') - \mathbf{v}(t)}{t' - t} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}.$$

It is a vector-valued function that accounts for both the rate of change of the magnitude and direction of velocity. Average acceleration and acceleration are both measured in **meters per second per second**, abbreviated "$\frac{\mathrm{m}}{\mathrm{s}^2}$".

Proposition 1.5.1 (Integral of acceleration is change in velocity)
Change in velocity is the integral of acceleration; that is,

$$\mathbf{v}(t_2) - \mathbf{v}(t_1) = \int_{t_1}^{t_2} \mathbf{a}(t) dt.$$

Proof. Since $\mathbf{a} = \frac{d\mathbf{v}}{dt}$, this follows from the Fundamental Theorem of Calculus. □

The next three formulae are equations typically learned in an introductory physics course.

Corollary 1.5.2 (Kinematic equations)
An object moving with constant acceleration $\mathbf{a}$ satisfies

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}(0) + \mathbf{a}t \\ \mathbf{x}(t) &= \mathbf{x}(0) + \mathbf{v}(0)t + \frac{1}{2}\mathbf{a}t^2 \\ v(t)^2 &= v(0)^2 + 2\mathbf{a} \cdot (\mathbf{x}(t) - \mathbf{x}(0)). \end{aligned}$$

Proof. The first equation follows from Proposition 1.5.1, since

$$\mathbf{v}(t) - \mathbf{v}(0) = \int_0^t \mathbf{a}\, dt = \mathbf{a}t.$$

The second equation follows from Proposition 1.3.1 and the first equation, since

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t \mathbf{v}(0) + \mathbf{a}t \, dt = \mathbf{v}(0)t + \frac{1}{2}\mathbf{a}t^2.$$

The third equation follows from the first two equations, since

$$\begin{aligned} v(0)^2 + 2\mathbf{a} \cdot (\mathbf{x}(t) - \mathbf{x}(0)) &= \mathbf{v}(0) \cdot \mathbf{v}(0) + 2\mathbf{a} \cdot \left(\mathbf{v}(0)t + \frac{1}{2}\mathbf{a}(0)t^2\right) \\ &= (\mathbf{v}(0) + \mathbf{a}(t)) \cdot (\mathbf{v}(0) + \mathbf{a}(t)) \\ &= v(t)^2. \end{aligned}$$

□

Abuse of Notation 1.5.3. While principally vector quantities, position, velocity, and acceleration are considered to be signed scalar quantities When dealing with one-dimensional motion.

> **Exercise 1.5.4.** An object starts at rest and its acceleration is constant. If it travels one meter in one second, determine:
>
> (a) the magnitude of its acceleration,
>
> (b) the magnitude of its velocity after one second, and
>
> (c) the magnitude of its displacement after two seconds.

§1.6 Tangential and centripetal acceleration

Prototypical example for this section: turning while driving.

The acceleration of a point particle can be written as the sum of a vector parallel to its velocity and the sum of a vector perpendicular to its velocity; these components are known as the particle's **tangential acceleration** and **centripetal acceleration**, respectively. Symbolically,

$$\mathbf{a}_T = \operatorname{proj}_{\mathbf{v}}(\mathbf{a})$$

and

$$\mathbf{a}_C = \mathbf{a} - \operatorname{proj}_{\mathbf{v}}(\mathbf{a}).$$

This distinction is useful because tangential acceleration changes a particle's speed, while centripetal acceleration changes a particle's direction.

> **Proposition 1.6.1** (Tangential acceleration is change in speed)
>
> The magnitude of a particle's tangential acceleration is equal to its change in speed; that is,
>
> $$\|\mathbf{a}_T\| = \frac{dv}{dt}.$$

Proof. Let $\hat{\mathbf{n}}(t)$ be a unit vector in the direction of the particle's velocity, so that

$$\mathbf{v} = v\hat{\mathbf{n}}.$$

Differentiating gives

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{n}} + v\frac{d\hat{\mathbf{n}}}{dt}.$$

Since $\hat{\mathbf{n}}$ has constant length,

$$0 = \frac{d}{dt}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) = 2\hat{\mathbf{n}} \cdot \frac{d\hat{\mathbf{n}}}{dt},$$

so $\frac{d\hat{\mathbf{n}}}{dt}$ is perpendicular to $\hat{\mathbf{n}}$. Hence, the component of $\mathbf{a}$ parallel to $\hat{\mathbf{n}}$ is $\frac{dv}{dt}\hat{\mathbf{n}}$, which has magnitude $\frac{dv}{dt}$. □

§1.7 Circular motion

Prototypical example for this section: a horse on a merry-go-round.

A point particle is in **circular motion** if its trajectory is restricted to a fixed circle. Its **angular speed** is defined to be the ratio of its speed to the radius of the circle; that is,

$$\omega = \pm\frac{v}{r}.$$

Angular speed can be thought of as a measure of rotation rate around the circle; it measures how fast the trajectory of a point particle sweeps out angle relative to the center of the circle. By convention, its sign is positive if the particle is moving counterclockwise with respect to some fixed orientation, and negative otherwise. The units of angular speed are $\frac{\text{rad}}{\text{s}} = \frac{1}{\text{s}}$.

Much like displacement, the **angular displacement** of a point particle with angular speed $\omega(t)$ between times t_1 and t_2 is the integral of its angular speed between times t_1 and t_2; that is,

$$\theta = \int_{t_1}^{t_2} \omega(t)dt.$$

It can be thought of the net angle through which the point particle rotates around the center during the time interval, where the sign convention is taken so that counterclockwise is positive. Angular displacement is measured in radians, which are unitless.

The **angular acceleration** of a point particle with angular speed $\omega(t)$ is the derivative of its angular speed; that is,

$$\alpha = \frac{d\omega}{dt}.$$

Proposition 1.7.1 (Integral of angular acceleration is change in angular velocity)
Change in angular velocity is the integral of angular acceleration; that is,

$$\omega(t_2) - \omega(t_1) = \int_{t_1}^{t_2} \alpha(t)dt.$$

Proof. Since $\alpha = \frac{d\omega}{dt}$, this follows from the Fundamental Theorem of Calculus. □

The kinematic equations also hold for the corresponding angular kinematic quantities.

Corollary 1.7.2 (Angular kinematic equations)
Let $\theta(t)$ be a function whose derivative is $\omega(t)$. An object in circular motion moving with constant angular acceleration α satisfies

$$\begin{aligned}\omega(t) &= \omega(0) + \alpha t \\ \theta(t) &= \theta(0) + \omega(0)t + \frac{1}{2}\alpha t^2 \\ \omega(t)^2 &= \omega(0)^2 + 2\alpha(\theta(t) - \theta(0)).\end{aligned}$$

Exercise 1.7.3. Prove this.

Remark 1.7.4 — The kinematic quantities associated with a particle in circular motion can also be applied to an object rotating around a fixed axis because the angular speed of every point on such a rotating object is the same.

The centripetal acceleration of a particle – that is, the component of acceleration that causes it to change direction – is easily calculable when in circular motion due to the following result.

Theorem 1.7.5 (Centripetal acceleration formula)
The centripetal acceleration of a point particle in circular motion has magnitude

$$\|\mathbf{a}_C\| = r\omega^2 = \frac{v^2}{r}.$$

Proof. Without loss of generality, assume the particle's trajectory is

$$\mathbf{x} = r\cos(\theta)\hat{\mathbf{i}} + r\sin(\theta)\hat{\mathbf{j}},$$

where $\theta(t)$ is a real-valued function whose derivative is $\omega(t)$; note that this makes

$$\mathbf{v} = -r\omega\sin(\theta)\hat{\mathbf{i}} + r\omega\cos(\theta)\hat{\mathbf{j}}.$$

Since acceleration is the second derivative of position,

$$\begin{aligned}\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left[r\omega(-\sin(\theta)\hat{\mathbf{i}} + \cos(\theta)\hat{\mathbf{j}})\right] \\ &= r\frac{d\omega}{dt}(-\sin(\theta)\hat{\mathbf{i}} + \cos(\theta) - r\omega^2(\cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}}) \\ &= r\frac{d\omega}{dt}\frac{\mathbf{v}}{r\omega} - r\omega^2(\cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}}).\end{aligned}$$

Hence the centripetal component of acceleration is $r\omega^2(\cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}})$, which has magnitude

$$r\omega^2 = r\left(\pm\frac{v}{r}\right)^2 = \frac{v^2}{r}.$$ □

The most common type of circular motion is uniform circular motion. A point particle is in **uniform circular motion** if its angular speed is constant.

The **period** of the particle in uniform circular motion is the time it takes for the particle to complete one revolution.

Corollary 1.7.6 (Period formula)
The period of a particle in uniform circular motion is

$$T = \frac{2\pi r}{v} = \frac{2\pi}{|\omega|}.$$

Proof. The length of the circumference of the circle is $2\pi r$, and its speed is always v. Since distance is the product of speed and time,

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{r\omega} = \frac{2\pi}{|\omega|}.$$ □

Exercise 1.7.7. In a certain country, the short hand of a clock is exactly half as long as the long hand, and rotates twice for each rotation of the long hand. The three points shown on the clock hands in Figure 1.1 have accelerations of magnitude a_A, a_B, and a_C. The point B is at the midpoint of the long hand. Rank a_A, a_B, and a_C from least to greatest.

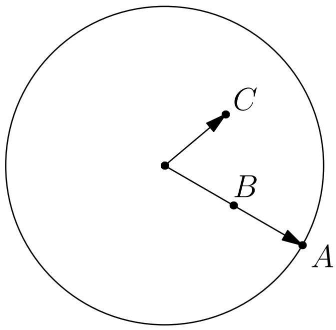

Figure 1.1: A special clock.

§1.8 Orbital angular velocity

Prototypical example for this section: the Earth orbiting around the Sun.

The **orbital angular velocity** of a point particle is the cross product of its position vector and its velocity vector divided by the square of its distance to the origin; that is,

$$\boldsymbol{\omega} = \frac{\mathbf{x}}{\|\mathbf{x}\|} \times \frac{\mathbf{v}}{\|\mathbf{x}\|} = \frac{\mathbf{x} \times \mathbf{v}}{\|\mathbf{x}\|^2}.$$

It can be thought of as a generalization of angular speed, since the particle is not restricted to circular motion. Indeed, the magnitude of the orbital angular velocity of a particle in circular motion around the origin is

$$\|\boldsymbol{\omega}\| = \left\| \frac{\mathbf{x} \times \mathbf{v}}{\|\mathbf{x}\|^2} \right\| = \frac{rv}{r^2} = |\omega|,$$

confirming the generalization

The orbital angular velocity vector encodes both the direction and rate at which a point particle sweeps out angle around the origin; it points perpendicular to both the position and velocity vectors. Orbital angular velocity is measured in **radians per second**, or $\frac{1}{\mathrm{s}}$.

Proposition 1.8.1

The magnitude of angular velocity is the rate at which the point particle sweeps out angle.

Proof. Let dt be an infinitesimally small time interval, and consider the area of the triangle formed by the three vectors $\mathbf{x}(t)$, $\mathbf{x}(t+dt)$, and $\mathbf{v}(t)dt$.

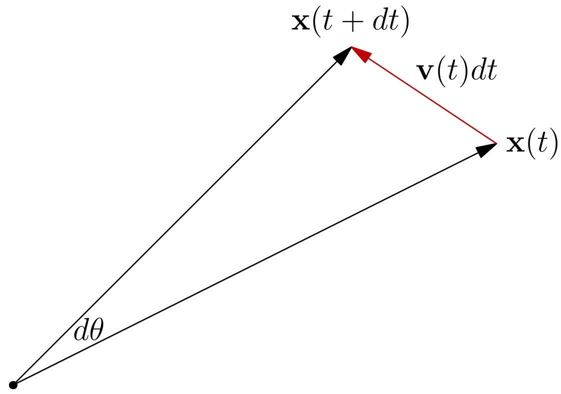

Figure 1.2: A particle moving over an infinitesimally small time interval.

The area of the triangle can be written in two ways, giving the equation

$$\frac{1}{2}\|\mathbf{x}(t)\times\mathbf{v}(t)dt\| = \frac{1}{2}\sin d\theta\,\|\mathbf{x}(t)\|\,\|\mathbf{x}(t+dt)\|,$$

where $d\theta$ is the angle between $\mathbf{x}(t)$ and $\mathbf{x}(t+dt)$. Since both dt and $d\theta$ are infinitesimally small, $\mathbf{x}(t+dt)=\mathbf{x}(t)$ and $\sin(d\theta)=d\theta$ up to first-order approximation. Solving for $\frac{d\theta}{dt}$ then gives

$$\frac{d\theta}{dt} = \frac{\|\mathbf{x}(t)\times\mathbf{v}(t)\|}{\|\mathbf{x}(t)\|^2} = \|\boldsymbol{\omega}\|. \qquad \square$$

§1.9 Spin angular velocity

Prototypical example for this section: the Earth rotating about its axis.

Since rotating rigid bodies cannot be modeled as point particles, the notion of orbital angular velocity for a rigid body is ill-defined. Instead, a different type of angular velocity, called spin angular velocity, is used; it measures the rate of change of the orientation of a rigid body without considering its translational movement.

The definition of spin angular velocity is motivated by noting that the velocity of every point particle in a two-dimensional object spinning in a fixed plane is proportional to and perpendicular to its position relative to the center of rotation; this constant of proportionality is the angular speed of the object. More generally, the velocity of every point particle in a rigid body spinning around a fixed axis is equal to the cross product of its relative position to some fixed point on the axis and some fixed vector. This vector is known as the rigid body's spin angular velocity. However, making this notion precise for a rigid body undergoing general motion is difficult. The next several paragraphs rigorously define and prove all of the properties of spin angular velocity that one would expect to be true.

Formally, the **spin angular velocity** of a rigid body with three non-collinear particles $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ is the vector

$$\boldsymbol{\omega} = -\frac{(\mathbf{v}_2 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_2)}{(\mathbf{x}_2 - \mathbf{x}_1) \cdot (\mathbf{v}_3 - \mathbf{v}_2)} = -\frac{(\mathbf{v}_1 - \mathbf{v}_2) \times (\mathbf{v}_3 - \mathbf{v}_2)}{(\mathbf{x}_1 - \mathbf{x}_2) \cdot (\mathbf{v}_3 - \mathbf{v}_2)}.$$

The denominator is zero when the particles are collinear, which makes sense because rotation is ambiguous in this case. In order for this definition to be rigorous, it needs to be shown that spin angular velocity is symmetric and also is independent of the triple of points chosen.

Proposition 1.9.1

The spin angular velocity expression is symmetric in its arguments, and the spin angular velocity of any three non-collinear particles in a rigid body is independent of the triple of points chosen.

Proof. First, it will be shown that the spin angular velocity expression is symmteric in $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$. Indeed, the numerator of the spin angular velocity expression is antisymmetric because it is a vector perpendicular to the plane of the triangle formed by $\mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_3 - \mathbf{v}_2$, and $\mathbf{v}_1 - \mathbf{v}_3$ with a magnitude equal to the triangle's area. The denominator of the spin angular velocity expression is also antisymmetric because

$$\begin{aligned}(\mathbf{x}_{i+1} - \mathbf{x}_i) \cdot (\mathbf{v}_{i+2} - \mathbf{v}_{i+1}) + (\mathbf{v}_{i+1} - \mathbf{v}_i) \cdot (\mathbf{x}_{i+2} - \mathbf{x}_{i+1}) &= \frac{d}{dt}(\mathbf{x}_{i+1} - \mathbf{x}_i) \cdot (\mathbf{x}_{i+2} - \mathbf{x}_{i+1}) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(\mathbf{x}_{i+1} - \mathbf{x}_i) \cdot (\mathbf{v}_{i+2} - \mathbf{v}_{i+1}) &- (\mathbf{x}_{i+2} - \mathbf{x}_{i+1}) \cdot (\mathbf{v}_i - \mathbf{v}_{i+2}) \\ &= (\mathbf{x}_{i+1} - \mathbf{x}_i) \cdot (\mathbf{v}_{i+2} - \mathbf{v}_{i+1}) + (\mathbf{v}_{i+2} - \mathbf{v}_{i+1}) \cdot (\mathbf{x}_i - \mathbf{x}_{i+2}) \\ &= -(\mathbf{v}_{i+2} - \mathbf{v}_i) \cdot (\mathbf{x}_{i+2} - \mathbf{x}_i) \\ &= -\frac{d}{dt}(\mathbf{x}_{i+2} - \mathbf{x}_i) \cdot (\mathbf{x}_{i+2} - \mathbf{x}_i) \\ &= 0,\end{aligned}$$

where indices cycle modulo 3. Therefore, spin angular velocity is symmetric upon any permutation of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

To show that spin angular velocity does not depend on the triple of points chosen, let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be three non-collinear points in a rigid body, and let $\mathbf{x}_i$ be any arbitrary point in the rigid body. It suffices to show that for any points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_i$ for which x_2 and x_i both aren't collinear with $\mathbf{x}_1$ and $\mathbf{x}_3$,

$$-\frac{(\mathbf{v}_1 - \mathbf{v}_2) \times (\mathbf{v}_3 - \mathbf{v}_2)}{(\mathbf{x}_1 - \mathbf{x}_2) \cdot (\mathbf{v}_3 - \mathbf{v}_2)} = -\frac{(\mathbf{v}_i - \mathbf{v}_2) \times (\mathbf{v}_3 - \mathbf{v}_2)}{(\mathbf{x}_i - \mathbf{x}_2) \cdot (\mathbf{v}_3 - \mathbf{v}_2)}.$$

Indeed, the numerator of their difference is

$$\begin{aligned}&[[(\mathbf{x}_2 - \mathbf{x}_i) \cdot (\mathbf{v}_3 - \mathbf{v}_2)]\,(\mathbf{v}_2 - \mathbf{v}_1) - [(\mathbf{x}_2 - \mathbf{x}_4) \cdot (\mathbf{v}_3 - \mathbf{v}_2)]\,(\mathbf{v}_2 - \mathbf{v}_i)] \times (\mathbf{v}_3 - \mathbf{v}_2) \\ &\quad = [(\mathbf{v}_3 - \mathbf{v}_2) \times [(\mathbf{v}_1 - \mathbf{v}_2) \times (\mathbf{v}_i - \mathbf{v}_2)]] \times (\mathbf{v}_3 - \mathbf{v}_2) \\ &\quad = 0,\end{aligned}$$

so the spin angular velocities of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ and $\{\mathbf{x}_i, \mathbf{x}_2, \mathbf{x}_3\}$ are equal. Repeating this process sufficiently many times shows that every pair of triangles within the rigid body has the same spin angular velocity. □

The following result verifies the desired property of spin angular velocity; that is, relative velocity is the cross product of spin angular velocity and relative position.

Corollary 1.9.2 (Angular velocity as cross product)
In any rigid body with spin angular velocity $\boldsymbol{\omega}$,

$$(\mathbf{v}_j - \mathbf{v}_i) = \boldsymbol{\omega} \times (\mathbf{x}_j - \mathbf{x}_i)$$

for all point particles $\mathbf{x}_i$ and $\mathbf{x}_j$ in the rigid body.

Proof. Let $\mathbf{x}$ be any point in the rigid body not on the line through $\mathbf{x}_i$ and $\mathbf{x}_j$. Then

$$\begin{aligned}
\boldsymbol{\omega} \times (\mathbf{x}_j - \mathbf{x}_i) &= -\frac{(\mathbf{v}_j - \mathbf{v}_i) \times (\mathbf{v} - \mathbf{v}_j)}{(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{v} - \mathbf{v}_j)} \times (\mathbf{x}_j - \mathbf{x}_i) \\
&= \frac{[(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{v} - \mathbf{v}_j)]\,(\mathbf{v}_j - \mathbf{v}_i) - [(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{v}_j - \mathbf{v}_i)]\,(\mathbf{v} - \mathbf{v}_j)}{(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{v} - \mathbf{v}_i)} \\
&= \frac{(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{v} - \mathbf{v}_j)}{(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{v} - \mathbf{v}_j)} (\mathbf{v}_j - \mathbf{v}_i) \\
&= \mathbf{v}_j - \mathbf{v}_i,
\end{aligned}$$

since $(\mathbf{x}_j - \mathbf{x}_i)$ is perpendicular to $\mathbf{v}_j - \mathbf{v}_i$ by differentiating $(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{x}_j - \mathbf{x}_i)$, which is constant. □

The spin angular velocity of an object does not depend any fixed origin; it is an intrinsic property of the object that does not change, even if the reference frame moves arbitrarily in a linear fashion. This is because the expression for spin angular velocity considers only relative positions and relative velocities between points in the rigid body.

The **spin angular speed** of a rigid body is a scalar quantity representing the magnitude of its spin angular velocity; that is,

$$\omega = \|\boldsymbol{\omega}\|.$$

It can be thought of as the instantaneous rate at which the orientation of an object changes, and is measured in radians per second.

Abuse of Notation 1.9.3. When the context is clear, spin angular velocity is abbreviated as just angular velocity, and spin angular speed is abbreviated as angular speed.

The desired properties of spin angular velocity can be verified to hold when considering fixed-axis rotation.

Proposition 1.9.4
If an object is rotating around a fixed axis ℓ, then

- its spin angular velocity is parallel to ℓ,
- its sign is determined by the right-hand rule, and
- its spin angular speed measures its angular speed around ℓ, in radians per second.

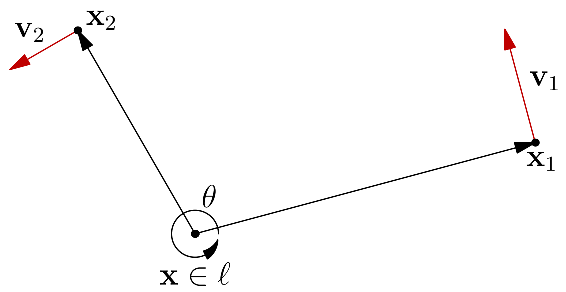

Figure 1.3: Two particles in a rigid body rotating around ℓ.

Proof. Let $\mathbf{x}$ be a fixed point on ℓ, and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be points in the plane through $\mathbf{x}$ perpendicular to ℓ. Additionally, let $\theta \notin \{0, \pi\}$ be the angle between $\mathbf{x}_1 - \mathbf{x}$ and $\mathbf{x}_2 - \mathbf{x}$. By properties of the cross product,

$$\boldsymbol{\omega} = -\frac{(\mathbf{v}_1 - \mathbf{v}) \times (\mathbf{v}_2 - \mathbf{v})}{(\mathbf{x}_1 - \mathbf{x}) \cdot (\mathbf{v}_2 - \mathbf{v})}$$

points in a direction perpendicular to the plane containing $\mathbf{x}$, $\mathbf{x}_1$, and $\mathbf{x}_2$ – that is, parallel to ℓ – and its sign points towards the vantage point that would make the rigid body appear to rotate counterclockwise, by the right-hand rule. Lastly, since the velocity vectors are perpendicular to the corresponding position vectors,

$$\omega = \|\boldsymbol{\omega}\| = \frac{\|(\mathbf{v}_1 - \mathbf{v}) \times (\mathbf{v}_2 - \mathbf{v})\|}{(\mathbf{x}_1 - \mathbf{x}) \cdot (\mathbf{v}_2 - \mathbf{v})} = \frac{\|\mathbf{v}_1\| \, \|\mathbf{v}_2\| \sin(\pi - \theta)}{\|\mathbf{x}_1\| \, \|\mathbf{v}_2\| \cos\left(\frac{\pi}{2} - \theta\right)} = \frac{\|\mathbf{v}_1\|}{\|\mathbf{x}_1\|},$$

which is exactly the angular speed around ℓ, in radians per second. □

Remark 1.9.5 — Even though the same symbol $\boldsymbol{\omega}$ is used for both orbital angular velocity and spin angular velocity, it is actually not so confusing, since orbital angular velocity is only defined for a point particle, while spin angular velocity is only defined for a rigid body. The same applies to angular acceleration, which is discussed in the next section.

§1.10 Spin angular acceleration

Prototypical example for this section: a bicycle tire when speeding up.

The **spin angular acceleration** of a rigid body is the derivative of its spin angular velocity with respect to time; that is,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}.$$

It can be thought of as the vector measure of the rate of change in both the axis around which the rigid body is rotating around and the rotation rate.

Proposition 1.10.1 (Integral of spin angular acceleration is change in spin angular velocity)
Change in spin angular velocity is the integral of spin angular acceleration; that is,

$$\boldsymbol{\omega}(t_2) - \boldsymbol{\omega}(t_1) = \int_{t_1}^{t_2} \boldsymbol{\alpha}(t)dt.$$

Proof. Since $\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$, this follows from the Fundamental Theorem of Calculus. □

Abuse of Notation 1.10.2. When the context is clear, spin angular acceleration is abbreviated as angular acceleration.

§1.11 Mass

Prototypical example for this section: any object.

The last fundamental kinematic quantity of a classical point particle is mass. Qualitatively, **mass** represents the amount of matter an object has; it is also a measure of its resistance to acceleration when pushed. Mass a fundamental scalar quantity measured in **kilograms**, abbreviated as "kg". The mass of a point particle is assumed to stay constant.

Remark 1.11.1 — A kilogram is approximately equal to the mass of $\frac{1}{1000}$ of a cubic meter of water.

Because the mass of point particles cannot change, the **law of conservation of mass** holds in a system that does not allow for the transfer of matter in and out of it.

Corollary 1.11.2 (Law of conservation of mass)
In a closed system, the total mass of all objects in the system is constant.

In other words, mass can neither be created nor destroyed.

§1.12 A few harder problems to think about

Problem 1A. The hard disk in a computer will spin up to speed within 10 rotations, but when turned off will spin through 50 rotations before coming to a stop. Assuming the hard disk has constant angular acceleration α_1 while speeding up and constant angular acceleration $-\alpha_2$ when slowing down, determine $\frac{\alpha_1}{\alpha_2}$.

Problem 1B. NASA trains astronauts to experience weightlessness with an airplane which flies in a parabolic arc with constant acceleration g toward the ground. The plane can remain on this trajectory for at most 25 seconds, due to the larger change in altitude required. If instead of simulating weightlessness, NASA wanted to fly a trajectory that would simulate the gravitational acceleration of Mars, which is $3.7\,\frac{\mathrm{m}}{\mathrm{s}^2}$, for what length of time can the place simulate Mars gravity?

Problem 1C. A train starts from city A and stops in city B. The distance between the cities is s. The train's maximal acceleration while speeding up is a_1 and its maximal acceleration while slowing down is $-a_2$. What is the shortest time in which the train can travel between A and B?

Problem 1D. The velocity versus position plot of a particle moving in a straight line is shown in Figure 1.4. Sketch the acceleration versus position plot.

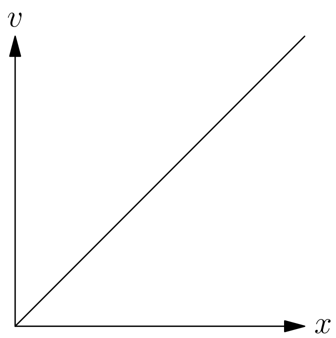

Figure 1.4: Velocity versus position.

Problem 1E. A disk of radius r rolls uniformly without slipping around the inside of a fixed hoop of radius R, as shown in Figure 1.5. The period the disc's motion around the hoop is T. Determine the instantaneous speed of the point on the disk opposite to the point of contact.

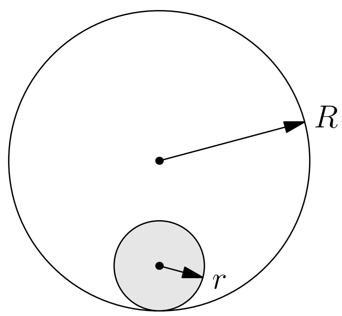

Figure 1.5: A disk rolling around a hoop.

Problem 1F. A train travels at 100 $\frac{\mathrm{m}}{\mathrm{s}}$ on an almost straight track. The track is slightly sinusoidal, with a vertical amplitude of h over a 1000 meter distance, as shown in Figure 1.6. If the maximum tolerable vertical acceleration of the train is set at 0.1 $\frac{\mathrm{m}}{\mathrm{s}^2}$, what is the maximum allowable size of h?

Problem 1G. A car is driving on a semicircular racetrack. Its velocity at several points along the track is shown in Figure 1.7. Sketch the car's acceleration at the same points.

Problem 1H. A wheel of radius r is rolling without slipping with angular speed ω. For point A on the wheel at an angle θ with respect to the vertical, shown in Figure 1.8, what is the magnitude of its velocity with respect to the ground?

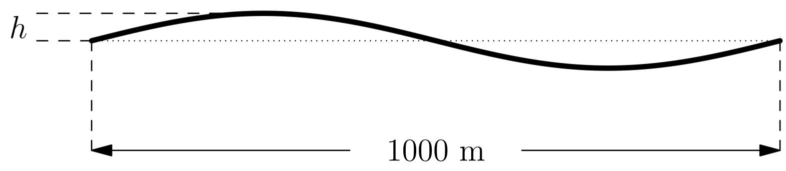

Figure 1.6: A slightly sinusoidal track.

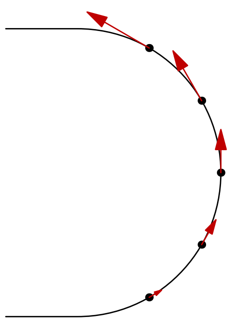

Figure 1.7: The velocity of a car driving on a semicircular racetrack.

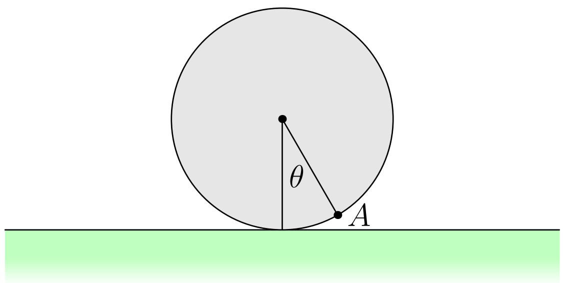

Figure 1.8: A rolling wheel.

2 Force

A **force** is any interaction that, when unopposed, will change the motion of an object; it can be thought of as a push or a pull. In this chapter, all objects are modeled as point particles. The **net force** on a point particle is the sum of all forces acting on the object; that is,

$$\mathbf{F} = \sum_i \mathbf{F}_i.$$

Force is a vector quantity measured in **newtons**, abbreviated "N".

> **Remark 2.0.1** — A newton is defined to be the magnitude of the force needed to accelerate a one-kilogram object at $1\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

From this, it follows that the newton can be expressed in terms of other base units:

$$1\,\mathrm{N} = 1\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}.$$

There are two broad categories of forces: **contact forces**, which occur between objects in contact with each other, and **non-contact forces**, which take place at a distance. With the exception of gravity, most visible interactions are due to contact forces. The six forces typically encountered in a macroscopic system are gravity, the normal force, friction, the tension force, the spring force, and drag.

§2.1 Gravity

Prototypical example for this section: a falling object.

Gravity is a non-contact force responsible for the tendency of objects to gravitate towards each other. **Newton's law of universal gravitation** quantifies the gravitational force two objects exert on each other.

Law 2.1.1 (Newton's law of universal gravitation)
There is a gravitational force between every pair of point particles whose magnitude is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The force that one object exerts on the other object points in the direction of the ray from the second object to the first object. The constant of proportionality is known as the **gravitational constant**, which is approximately

$$G \approx 6.674 \times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}.$$

Symbolically, Newton's law of universal gravitation asserts that the gravitational force between two objects has magnitude

$$\|\mathbf{F}\| = G\frac{m_1 m_2}{r^2},$$

where m_1 and m_2 are the masses of the two objects and r is the distance between them.

Remark 2.1.2 — Because the gravitational constant is small, the gravitational force between two objects with small mass can be neglected and only the gravity exerted by celestial bodies needs to be considered.

The magnitude of the gravitational force exerted on an object by the planet it is on is the object's **weight**, measured in newtons.

The **Earth** is the planet humanity lives on and can be modeled as a uniform spherical object with radius 6.371×10^6 m and mass 5.972×10^{24} kg. Newton's law of universal gravitation can be used to determine the force the Earth exerts on objects near its surface.

Corollary 2.1.3 (Gravity on Earth)
The force the Earth exerts on small objects near its surface has a magnitude proportional to the object's mass, and this force points towards the center of the Earth. This constant of proportionality is the **gravity of Earth**, which is approximately

$$g \approx 9.807 \, \frac{\mathrm{m}}{\mathrm{s}^2}.$$

Symbolically,

$$\|\mathbf{F}\| = mg.$$

Proof. By the shell theorem (Theorem 8.2.1), the Earth can be treated as a point particle, so the force exerted on the object by the Earth is approximately

$$\|\mathbf{F}\| \approx \left(6.674 \times 10^{-11} \, \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}\right) \cdot \frac{(5.972 \times 10^{24} \, \mathrm{kg}) \cdot m}{(6.371 \times 10^6 \, \mathrm{m})^2} \approx 9.807 \, \frac{\mathrm{m}}{\mathrm{s}^2} \cdot m = mg,$$

by Newton's law of universal gravitation. □

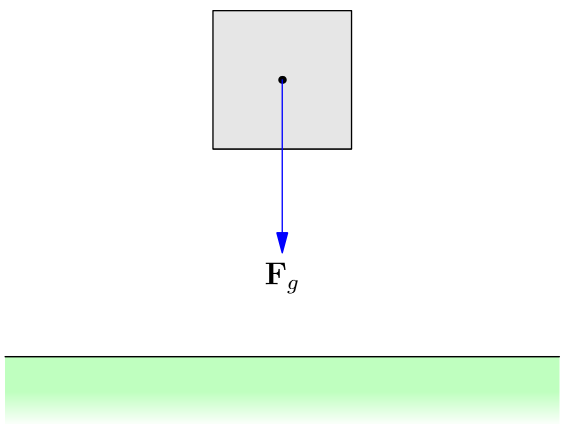

Figure 2.1: The gravitational force exerted on the block by the Earth.

In this text, all non-celestial systems are assumed to take place on Earth unless stated otherwise.

§2.2 Normal force

Prototypical example for this section: a block on a ramp.

The **normal force** exerted by an object in contact with another object is the force which prevents them from passing through each other. It is a contact force, and it is perpendicular to the plane of contact between the two objects because motion parallel to the surface of contact is not interpenetrating.

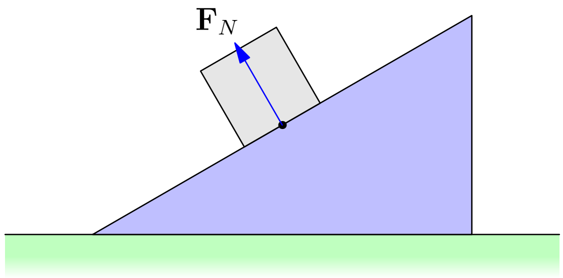

Figure 2.2: The normal force exerted on the block by the ramp.

A **spring scale** measures the normal force acting on the top of it.

> **Exercise 2.2.1.** A car collides into a wall. Characterize the forces acting on the car.

> **Exercise 2.2.2.** Two balls collide at an oblique angle. Characterize all the forces acting on each ball.

§2.3 Friction

Prototypical example for this section: a stationary block on a ramp.

Friction is the force resisting the relative motion of two objects sliding against each other. It is a contact force, and it always points in the direction that opposes movement or potential movement. Because the frictional force is parallel to the plane of contact between the two objects, it must be perpendicular to the normal force between the two objects.

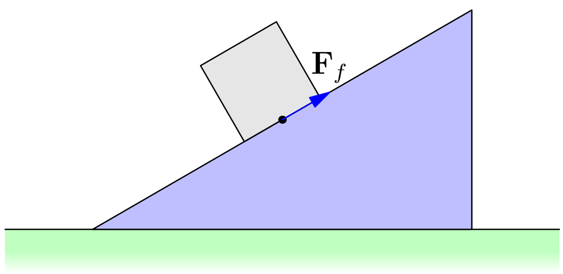

Figure 2.3: The frictional force exerted on the block by the ramp.

There are two kinds of friction: **kinetic friction**, which is friction between objects moving relative to each other, and **static friction**, which is friction between objects not moving relative to each other. **Amonton's laws of friction** govern the relationship between the force of kinetic friction and the normal force, and the **Coulomb friction law** governs the relationship between the force of static friction and the normal force.

Law 2.3.1 (Amontons' laws of friction)

The magnitude of kinetic friction between two objects moving relative to each other is

(a) directly proportional to the normal force between the objects (1st law),

(b) independent of the contact area (2nd law), and

(c) independent of the sliding velocity (Coulomb's law of dry friction).

The constant of proportionality in Amontons' 1st law is the **coefficient of kinetic friction** μ_{k} between the two surfaces. It is a dimensionless constant that depends only on the material composition of the two objects. Symbolically,

$$\|\mathbf{F}_f\| = \mu_{\mathrm{k}} \|\mathbf{F}_N\| .$$

The coefficient of kinetic friction between two objects is typically between 0.3 and 0.6.

Law 2.3.2 (Coulomb friction law)

The maximum magnitude of static friction between two objects not sliding relative to each other is proportional to the normal force. The constant of proportionality is the **coefficient of static friction** μ_{s} between the two surfaces. It is a dimensionless constant that depends only on the material composition of the two objects. Symbolically,

$$\|\mathbf{F}_f\| \leq \mu_{\mathrm{s}} \|\mathbf{F}_N\| .$$

The following relationship between the coefficient of static friction and the coefficient of kinetic friction always holds:

Corollary 2.3.3 (Static friction is at least kinetic friction)

The coefficient of static friction between two objects is always at least the coefficient of kinetic friction between them; that is,

$$\mu_{\mathrm{k}} \leq \mu_{\mathrm{s}}.$$

Proof. Suppose the coefficient of static friction is less than the coefficient of kinetic friction. Then the force required to keep the objects in motion relative to each other would be greater than the force required to start the motion, which is absurd. □

§2.4 Tension force

Prototypical example for this section: a block hanging from a string.

The **tension force** is a contact force transmitted by a string. The **tension** of a string, measured in newtons, is defined to be the magnitude of the tension force it transmits.

Ideal strings have no mass, fixed length, and uniform tension throughout. All strings are assumed to be ideal unless otherwise specified. The tension force points towards the string.

Strings can curve around a **pulley**, which direct strings in a particular direction. Ideal pulleys are frictionless and massless.

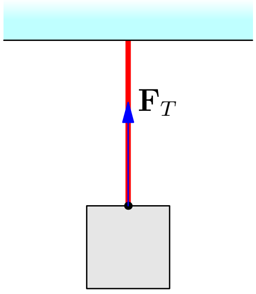

Figure 2.4: The tension force exerted on the block by the string.

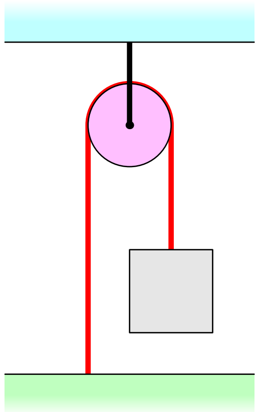

Figure 2.5: A string wrapped around a pulley.

> **Exercise 2.4.1.** Two people standing on Earth are pulling on opposite ends of a rope. Characterize all forces acting on each person.

§2.5 Spring force

Prototypical example for this section: a block hanging from a spring.

The **spring force** is the contact force exerted by a spring on an object directed towards the spring's **equilibrium position**, or the position at which the spring exerts no force. Ideal springs are one-dimensional and have no mass.

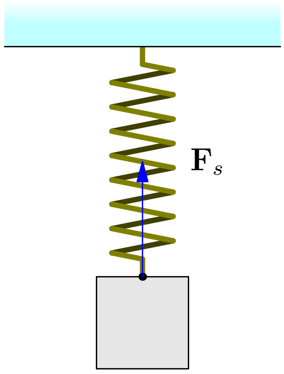

Figure 2.6: The spring force exerted on the block by the spring.

Hooke's law quantifies the magnitude of the spring force for ideal springs.

Law 2.5.1 (Hooke's law)

An ideal spring whose displacement from equilibrium position is $\mathbf{x}$ exerts a force proportional to and in the opposite direction of $\mathbf{x}$. The constant of proportionality is the spring's **spring constant** k. The spring constant is a property of the spring, and its units are $\frac{\text{N}}{\text{m}} = \frac{\text{kg}}{\text{s}^2}$. Symbolically,

$$\mathbf{F}_s = -k\mathbf{x}.$$

The spring constant of a spring is typically between $10\ \frac{\text{N}}{\text{m}}$ and $10^5\ \frac{\text{N}}{\text{m}}$.

Exercise 2.5.2. Two blocks connected by a spring hang from two strings as shown in Figure 2.7. Characterize the forces acting on each block.

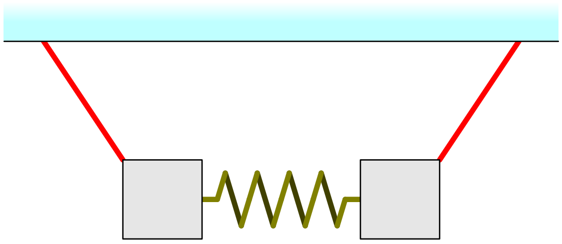

Figure 2.7: Two hanging blocks connected by a spring.

§2.6 Drag

Prototypical example for this section: air resistance.

Drag is a contact force acting on an object traveling through a fluid, such as air or water. It is a contact force and always points in the direction opposite the relative motion of the object through the fluid. The **drag equation** quantifies the magnitude of the drag force under ideal behavior.

Law 2.6.1 (Drag equation)

The magnitude of the drag force experienced on an object is proportional to the density of the fluid, proportional to square of the speed of the object relative to the fluid, and proportional to the reference area. Symbolically,

$$\|\mathbf{F}_D\| = \frac{1}{2}\rho v^2 C_D A,$$

where ρ is the density of the fluid, v is the speed of the object relative to the fluid, and A is the reference area, or frontal area. where $\frac{1}{2}C_D$ is the constant of proportionality. C_D is known as **drag coefficient**, a dimensionless quantity that depends on both the object's geometry and the type of fluid.

Conventionally, the drag force from air resistance is ignored unless otherwise stated.

§2.7 Buoyant force

Prototypical example for this section: a boat floating on water.

The **buoyant force** is a contact force exerted by a fluid that opposes the weight of a partially or fully immersed object. **Archimedes' principle** quantifies the magnitude of the buoyant force on an object.

Theorem 2.7.1 (Archimedes' principle)
The buoyant force exerted on an object in a fluid is equal in magnitude and opposite in direction to the weight of the displaced fluid.

Archimedes' principle is proved in Part **??**.

Generally, the buoyant force due to air is ignored unless otherwise stated, because the mass of displaced air is usually neglectable.

§2.8 Newton's third law

Newton's third law states that all forces occur in pairs.

Law 2.8.1
If point particle A exerts a force $\mathbf{F}$ on point particle B, then B must exert a force $-\mathbf{F}$ on A. In other words, the forces are equal in magnitude and opposite in direction. Additionally, the force must in the direction of the line connecting A and B.

By Exercise 3.3.3, Newton's third law also holds for objects and not just point particles. As expected, Newton's third law applies to each of the five types of forces described above. For example,

- the gravitational force exerted by the Sun on the Earth is equal in magnitude and opposite in direction to the gravitational force exerted by the Earth on the Sun,
- the normal force exerted by a block on a table is equal in magnitude and opposite in direction to the normal force the table exerts on the block,
- the frictional force a shoe exerts on the ground is equal in magnitude and opposite in direction to the force the ground exerts on the shoe,
- the tension force exerted on the one team in a game of tug of war is equal in magnitude and opposite in direction to the tension force exerted on the other team, and
- the spring force exerted on a block by a spring hanging from a ceiling is equal in magnitude and opposite in direction to the spring force exerted on the ceiling by the hanging block.

For strings that curve around a pulley, the equal and opposite force is exerted on the pulley.

§2.9 Reference frames

Prototypical example for this section: the coordinate system fixing the Earth.

All kinematic quantities discussed in Chapter 1 have been measured relative to the absolute coordinate system $V = \mathbb{R}^3$ defined in the beginning of this chapter. However, it is often useful to consider coordinate systems that may continuously vary over time relative to V. This moving coordinate system is known as a **frame of reference**. The kinematic quantities change depending on what frame of reference is used, so when no frame of reference is specified, all non-celestial systems are assumed to take place in the frame of reference fixing the Earth.

Example 2.9.1 (Train)

Consider a train accelerating away from a station. If the frame of reference fixes the train, then the entire world accelerates away from the train. On the other hand, the train simply accelerates away from the station in the frame of reference fixing the Earth.

Example 2.9.2 (Merry-go-round)

Consider a rotating merry-go-round. In the frame of reference fixing the merry-go-round, then the entire world rotates around the merry-go-round. On the other hand, the merry-go-round simply rotates around its axis in the frame of reference fixing the Earth.

In general, it is often useful to consider a reference frame fixing an object.

§2.10 Newton's first law

Newton's first law asserts the existence of inertial reference frames.

Law 2.10.1 (Newton's first law of motion)

There exists a frame of reference where every object moves with constant velocity if and only if there is no net force acting on it. Such a frame is called an **inertial reference frame**.

In other words, Newton's first law states that in an inertial reference frame, an object at rest will stay at rest and an object in motion will stay in motion unless acted on by a net force. Symbolically,

$$\mathbf{F} = 0 \iff \frac{d\mathbf{v}}{dt} = 0.$$

Because of Newton's first law, all systems in classical mechanics are assumed to take place in inertial reference frames unless otherwise specified.

Example 2.10.2 (Accelerating reference frame)

Consider a train accelerating away from a station. The reference frame fixing the train is *not* inertial because objects in the train appear to have the tendency to accelerate towards the back of the train; for example, a cup on an accelerating train will has the tendency to move backwards. However, the reference frame fixing the station *is* inertial because the apparent backwards acceleration is a result of an object's tendency to move with constant velocity in a reference frame fixing the ground.

Example 2.10.3 (Rotating reference frame)

Consider a rotating merry-go-round. The reference frame fixing the merry-go-round is *not* inertial because objects near the edge of the merry-go-round appear have the tendency to accelerate outwards. However, the reference frame fixing the ground *is* inertial because the apparent outward acceleration is a result of an object's tendency to move in a straight line in a reference frame fixing the ground.

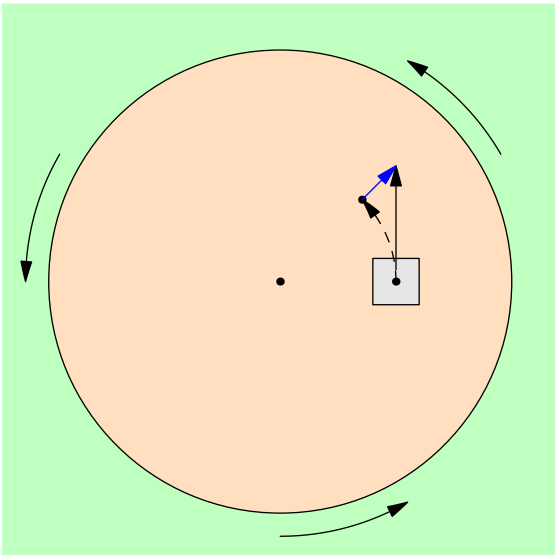

Figure 2.8: An apparent outwards acceleration in a rotating reference frame.

Strictly speaking, the reference frame fixing the Earth is *not* an inertial reference frame because the Earth rotates. However, deviations are minuscule, so the Earth can be considered an inertial reference frame for most practical purposes.

Exercise 2.10.4. Explain why the reference frame constructed by taking an inertial reference frame and moving it at a constant velocity is also inertial.

§2.11 A few harder problems to think about

Problem 2A. Three blocks of masses m_1, m_2, and m_3 are stacked on top of each other as shown in Figure 2.9. Determine the magnitude of the normal force that each pair of adjacent blocks exerts on each other.

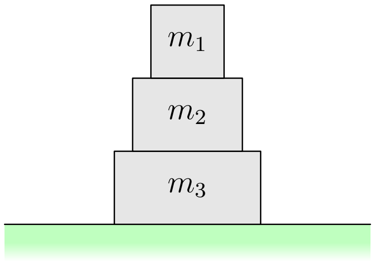

Figure 2.9: Three stacked blocks.

Problem 2B. Three blocks of masses m_1, m_2, and m_3 are hanging from three strings as shown in Figure 2.10. Determine the tension of each string.

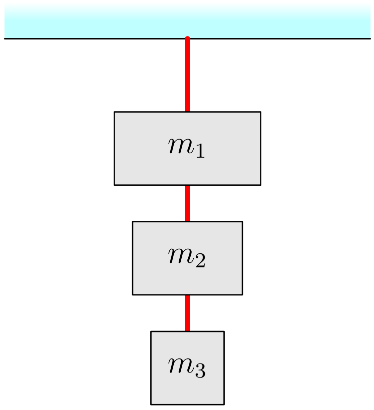

Figure 2.10: Three hanging blocks.

Problem 2C. A scale is calibrated so that it gives a correct reading when sitting on the ground. A person holds the scale and presses it on both sides with their hands pushing up on the bottom with the left hand and pushing down on the top with the right. The scale has a mass of 5 kg, and their left hand exerts a force of 200 N. What is the reading on the scale?

Problem 2D. A block is sliding down an inclined plane with inclination angle α at a constant velocity as shown in Figure 2.11. Determine the coefficient of kinetic friction between the block and the inclined plane.

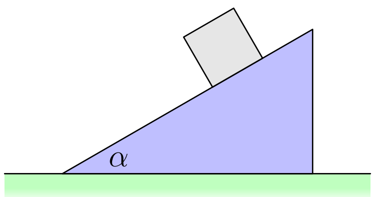

Figure 2.11: A block sliding down an inclined plane.

Problem 2E. Three cubical blocks of the same volume are made out of wood, Styrofoam, and plastic. When the plastic block is placed in water, half of its volume is submerged. If the wooden block is placed in water with the plastic block on top, the wooden block is just fully submerged. Similarly, if the Styrofoam block is placed in oil with the plastic block on top, the Styrofoam block is just fully submerged. The density of oil is 0.7 times that of the water. What is the ratio of the density of wood to the density of Styrofoam?

Problem 2F. A block with mass m is hanging from two strings that make angles of $0 < \alpha < 90°$ and $0 < \beta < 90°$ with the ceiling as shown in Figure 2.12. Determine the tension of each string.

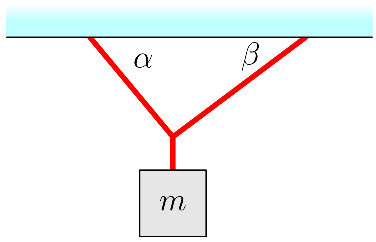

Figure 2.12: A block hanging from two strings.

Problem 2G. Two equal masses m are connected by an elastic string that acts like an ideal spring with spring constant k and unstretched length l. The two masses are hung over a pulley, as shown in Figure 2.13. Determine the total length of the string at equilibrium.

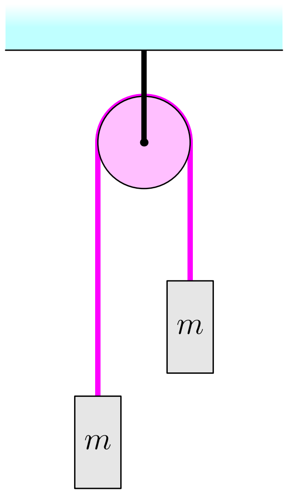

Figure 2.13: Two masses hanging from an elastic string.

Problem 2H. A block of mass m is attached to a massless string. The string is passed over a massless pulley and the end of the string is fixed in place. The horizontal part of the string has length l. Now a small mass m is hung from the horizontal part of the string, and the system comes to equilibrium, as shown in Figure 2.14.

(a) Determine the tension at the end of the string.

(b) Determine the height by which the block is raised. Assume the pulley has neglectable size.

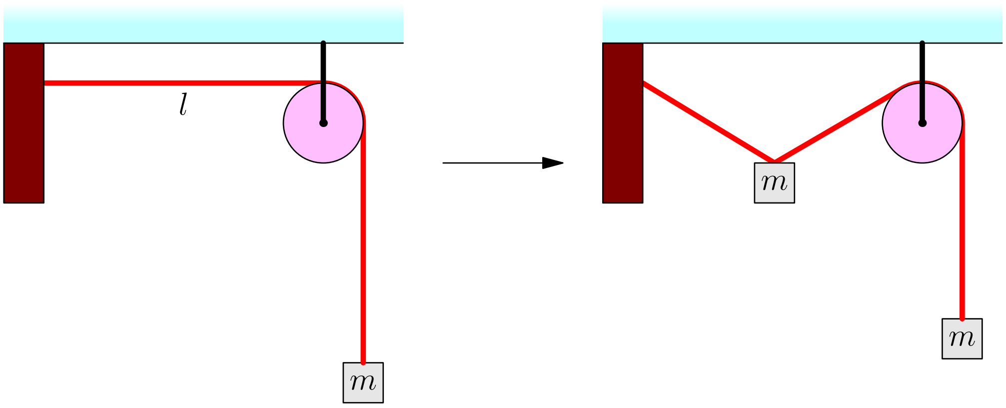

Figure 2.14: A mass is placed in the middle of the horizontal string.

Problem 2I. Three blocks, five strings, and five pulleys hang as shown in Figure 2.15. The center block has a mass of 1 kg. Determine the mass of the other two blocks.

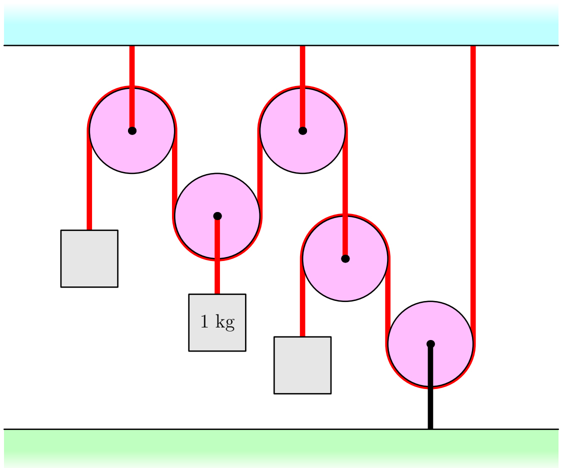

Figure 2.15: Three blocks, five strings, and five pulleys.

3 Linear dynamics

Dynamics is the study of forces and their effects on the motion of objects. **Linear dynamics** studies the linear, non-rotating component of an object's motion. In this chapter, all objects are modeled as rigid bodies.

§3.1 Systems

Prototypical example for this section: the solar system, balls on a billiard table, or any rigid body.

A **system** is a collection of point particles; in particular, recall that a rigid body is a system in which the point particles do not move relative to each other. The **mass** of a system is the sum of the masses of the point particles comprising the system:

$$m = \sum_i m_i.$$

The **center of mass** of a system is the weighted sum of the positions of the point particles:

$$\mathbf{x}_{\text{cm}} = \sum_i \left[\frac{m_i}{\sum_i m_i} \cdot \mathbf{x}_i \right] = \frac{\sum_i m_i \mathbf{x}_i}{m}.$$

The **linear velocity** of a system, often abbreviated as just **velocity**, is the derivative of the position of its center of mass:

$$\mathbf{v}_{\text{cm}} = \frac{d\mathbf{x}_{\text{cm}}}{dt}.$$

Lastly, the **acceleration** of a system is the derivative of its velocity:

$$\mathbf{a}_{\text{cm}} = \frac{d\mathbf{v}_{\text{cm}}}{dt}.$$

Abuse of Notation 3.1.1. When the context is clear, the subscripts in $\mathbf{x}_{\text{cm}}$, $\mathbf{v}_{\text{cm}}$, and $\mathbf{a}_{\text{cm}}$ are often omitted, and center of mass, velocity, and acceleration are abbreviated as $\mathbf{x}$, $\mathbf{v}$, and $\mathbf{a}$.

It is often useful to consider the center of mass, linear velocity, and acceleration of a rigid body because these quantities dictate the non-rotating motion of the object.

§3.2 Momentum

Prototypical example for this section: a moving car.

The **momentum** of a point particle is a vector equal to the product of its mass and velocity; that is,

$$\mathbf{p} = m\mathbf{v}.$$

The **momentum** of a system is a vector equal to the sum of the momenta of the point particles that comprise the system; that is,

$$\mathbf{p} = \sum_i m_i \mathbf{v}_i.$$

The units of momentum are $\frac{\text{kg}\cdot\text{m}}{\text{s}}$, the product of the units for mass and velocity; these units are also the same as $\text{N}\cdot\text{s}$. Momentum can be thought of as the quantity of motion.

If the velocity of a system's center of mass is known, the momentum of a system can be calculated.

Corollary 3.2.1

The momentum of an system is equal to the product of its mass and the velocity of its center of mass; that is,

$$\mathbf{p} = m\mathbf{v}.$$

Proof. By the definition of center of mass,

$$\mathbf{p} = \sum_i m_i \mathbf{v}_i = \sum_i m_i \frac{d\mathbf{x}_i}{dt} = \frac{d}{dt} \sum_i m_i \mathbf{x}_i = \frac{d}{dt} m\mathbf{x} = m\mathbf{v}. \qquad \square$$

§3.3 Newton's second law

Newton's second law equates the net force on a point particle with the rate of change of its momentum.

Law 3.3.1 (Newton's second law of motion)

In an inertial reference frame, the instantaneous change in momentum of a point particle is equal to the net force acting on the point particle; that is,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

Newton's second law can be extended to systems by defining the **net force** on a system is the sum of all forces acting on the point particles that comprise the system, regardless of the locations the forces are applied to. This extension is known as **Euler's first law**.

Theorem 3.3.2 (Euler's first law of motion)

In an inertial reference frame, the change in momentum of a system is equal to the net external force acting on the system; that is,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

Proof. Let $\mathbf{F}_i$ be the (possibly zero) external force acting on each point particle i in the system, and let particle i exert a force $\mathbf{F}_{ij}$ on particle j. Then the net force acting on the rigid body is

$$\mathbf{F} = \sum_i \mathbf{F}_i = \sum_i [\mathbf{F}_i] + \sum_{i\neq j} [\mathbf{F}_{ij}] = \sum_i \left[\mathbf{F}_i + \sum_{j\neq i} \mathbf{F}_{ji}\right] = \sum_i \frac{d\mathbf{p}_i}{dt} = \frac{d}{dt} \sum_i \mathbf{p}_i = \frac{d\mathbf{p}}{dt},$$

where the second equality follows because $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ by Newton's third law, and the fourth equality follows because the net force acting on a particle is its change in momentum by Newton's third law. □

Exercise 3.3.3. Prove that Newton's third law can also be extended to rigid bodies; that is, the net force that rigid body A exerts on rigid body B is equal in magnitude and opposite in direction to the net force B exerts on A.

If a system's mass is constant, Euler's first law associates the net force on an system with its acceleration.

Theorem 3.3.4 (F=ma)

In an inertial reference frame, the net force acting on an system with constant mass is the product of its mass and its acceleration; that is,

$$\mathbf{F} = m\mathbf{a}.$$

Proof. Since mass is constant, Euler's first law gives

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}.$$ □

Question 3.3.5. Team A and team B are playing tug-of-war. By Newton's third law, the force exerted on team A by team B is equal to the force exerted on team B by team A. However, Theorem 3.3.4 states that a nonzero net force needs to be present for someone to win, or else the rope can never accelerate. How is this possible for a team to win?

Much like the law of conservation of mass, the **law of conservation of momentum** holds in a closed system not acted upon by external forces. It is a corollary of Newton's third law and Euler's first law.

Theorem 3.3.6 (Law of conservation of momentum)

In a closed system, the total momentum is constant.

Proof. In a closed system, the net force acting on the system is zero. Since

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \mathbf{0},$$

momentum is constant. □

Question 3.3.7. An object is accelerating towards the Earth, increasing its momentum. Why does this not violate conservation of momentum?

§3.4 Impulse

Prototypical example for this section: A car crashing into a wall.

The **impulse** of a force over a time interval is the integral of the force over the time interval; that is,

$$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F}\, dt.$$

It is a vector quantity measured in $\mathrm{N} \cdot \mathrm{s}$, the product of the units for mass and velocity; these units are also the same as $\frac{\mathrm{kg} \cdot \mathrm{m}}{\mathrm{s}}$.

The **impulse-momentum theorem** states that the impulse of the net force on an object is equal to its change in momentum:

Theorem 3.4.1 (Impulse-momentum theorem)
The impulse of a net force on an object over a time period is equal to its change in momentum over that time period; that is,

$$\mathbf{J} = \mathbf{p}_2 - \mathbf{p}_1.$$

Proof. By Euler's first law and the Fundamental Theorem of Calculus,

$$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F} dt = \int_{t_1}^{t_2} \frac{d\mathbf{p}}{dt} dt = \mathbf{p}_2 - \mathbf{p}_1. \qquad \square$$

§3.5 Projectile motion

Prototypical example for this section: a cannonball shot from a cannon.

A **projectile** is any object that is thrown such that the only force acting on it throughout its trajectory is gravity. In particular, air resistance is ignored.

Theorem 3.5.1 (Projectile motion)
A projectile with initial position $\mathbf{x}_0$ and initial velocity $\mathbf{v}_0$ has trajectory

$$\mathbf{x} = -\frac{1}{2} g \hat{\mathbf{k}} t^2 + \mathbf{v}_0 t + \mathbf{x}_0.$$

Proof. By Newton's second law,

$$\frac{d^2\mathbf{x}}{dt^2} = \mathbf{a} = \frac{\mathbf{F}}{m} = \frac{-mg\hat{\mathbf{k}}}{m} = -g\hat{\mathbf{k}},$$

so the projectile undergoes constant acceleration and thus has trajectory formula given by Corollary 1.5.2. $\square$

Exercise 3.5.2. An object is dropped from rest at a height of h above the ground. Determine the time it takes for it to hit the ground, as well as its speed at the moment of impact.

§3.6 Atwood machine

Prototypical example for this section: an elevator.

An **Atwood machine** is a system in which two objects connected by a string are hanging from a pulley.

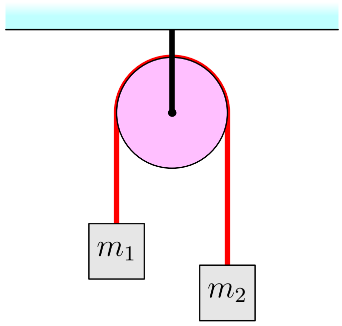

Figure 3.1: An Atwood machine.

Theorem 3.6.1 (Atwood machine formulae)
The tension of the string in an Atwood machine is

$$\frac{2gm_1m_2}{m_1+m_2} = \frac{2g}{\frac{1}{m_1}+\frac{1}{m_2}}$$

and the magnitude of the acceleration of the blocks in an Atwood machine is

$$g\frac{|m_1-m_2|}{m_1+m_2},$$

where g is Earth's gravity.

Proof. Let F_T be the tension in the string. Since the weight of the objects are gm_1 and gm_2, the magnitude of the net force acting on the m_1 is $|gm_1 - F_T|$ and the magnitude of the net force acting on the m_2 is $|gm_2 - F_T|$. Since the accelerations of both blocks have equal magnitudes and opposite directions,

$$a = \frac{gm_1 - F_T}{m_1} = -\frac{gm_2 - F_T}{m_2},$$

by $\mathbf{F} = m\mathbf{a}$. Solving for F_T gives

$$F_T = \frac{2gm_1m_2}{m_1+m_2}.$$

Rewriting the acceleration equations as $am_1 = gm_1 - F_T$ and $am_2 = F_T - gm_2$ gives

$$|a| = g\frac{|m_1-m_2|}{m_1+m_2}$$

by adding the equations. $\square$

§3.7 Terminal velocity

Prototypical example for this section: a skydiver falling at maximum speed.

The **terminal velocity** of an object is the maximum speed that can be attained by an object falling through a fluid, such as air.

Theorem 3.7.1 (Terminal velocity formula)
The terminal velocity of an object falling towards Earth is approximately

$$v_t \approx \sqrt{\frac{2mg}{\rho C_d A}},$$

where g is Earth's gravity, ρ is the density of the air, A is the reference area, and C_D is the drag coefficient.

Proof. When an object is at its terminal velocity v_t, it is moving at a constant velocity, so the net force acting on the object is zero. Since the buoyant force is neglectable, this means the magnitude of its weight approximately equals the magnitude of its drag force:

$$mg \approx \frac{1}{2}\rho v_t^2 C_D A.$$

Solving for v gives

$$v_t \approx \sqrt{\frac{2mg}{\rho C_D A}}.$$

□

§3.8 Tsiolkovsky rocket equation

Tsiolkovsky's rocket equation gives a way to determine the increase in a rocket's velocity through fuel ejection. It assumes that no other forces, such as gravity, act on the rocket.

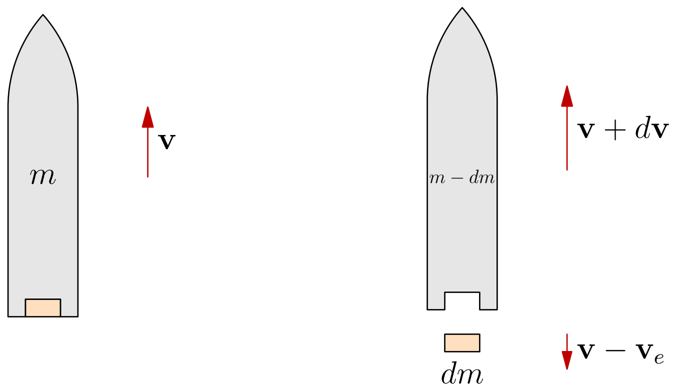

Figure 3.2: A rocket ejecting dm fuel over dt time.

Theorem 3.8.1 (Tsiolkovsky rocket equation)
A rocket with initial mass m_0 is ejecting fuel at a constant rate and constant velocity $\mathbf{v}_e$ relative to the rocket, ending its journey with final mass m_f. If no other forces act on the rocket, its change in velocity is

$$\mathbf{v}_e \ln \frac{m_0}{m_f}.$$

Proof. Let $\mathbf{v}$ be the velocity of the rocket, and suppose an infinitesimally small mass of fuel dm is ejected from the rocket over a time interval dt, increasing the rocket's velocity by $d\mathbf{v}$.

Since the absolute velocity of the fuel is $\mathbf{v} - \mathbf{v}_e$, conservation of momentum gives

$$m\mathbf{v} = (m - dm)(\mathbf{v} + d\mathbf{v}) + (\mathbf{v} - \mathbf{v}_e)dm.$$

Expansion and division by dt gives

$$\begin{aligned} \mathbf{v}_e \cdot dm &= m \cdot d\mathbf{v} - dm \cdot d\mathbf{v} \\ \frac{d\mathbf{v}}{dt} &= \frac{\mathbf{v}_e}{m}\frac{dm}{dt} \end{aligned}$$

since $dm \cdot dv$ is a second-order term that vanishes. Finally, integrating both sides over the duration of fuel ejection gives

$$\begin{aligned} \int_{\mathbf{v}(0)}^{\mathbf{v}(t_f)} \frac{d\mathbf{v}}{dt} &= \int_{m_0}^{m_f} \frac{\mathbf{v}_e}{m}\frac{dm}{dt} \\ \mathbf{v}(t_f) - \mathbf{v}(0) &= \mathbf{v}_e(\ln m_f - \ln m_0) = \mathbf{v}_e \ln \frac{m_f}{m_0}, \end{aligned}$$

where $\mathbf{v}(0)$ is the initial velocity of the rocket and $\mathbf{v}(t_f)$ is the final velocity of the rocket. □

§3.9 A few harder problems to think about

Problem 3A. A car with mass m is traveling at a speed of $\mathbf{v}$ before it starts applying its brakes, locking its tires in place. In terms of m, $\mathbf{v}$, and the coefficient of kinetic friction μ_k between the tires and the road, determine:

(a) the amount of time it takes for the car to stop, and

(b) the distance it travels before it stops.

Problem 3B. A block with mass m_1 and velocity $\mathbf{v}_1$ collides with a block with mass m_2 and velocity $\mathbf{v}_2$, as shown in Figure 3.3, and the blocks stick together. Find the velocity of the blocks after colliding and the impulse each block experiences. Assume there is no friction between the ground and the blocks.

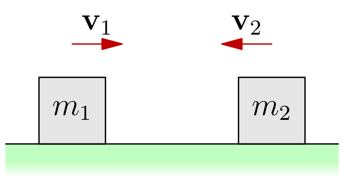

Figure 3.3: Colliding blocks.

Problem 3C. Projectiles A, B, and C are simultaneously thrown off a cliff and take the trajectories shown in Figure 3.4. Rank the times they take to hit the ground.

Problem 3D. Two blocks of masses $m_1 = 2.0\,\text{kg}$ and $m_2 = 1.0\,\text{kg}$ are stacked together on top of a frictionless table as shown in Figure 3.5. The coefficient of static friction between the blocks is $\mu_s = 0.20$. Determine the minimum magnitude of the horizontal force that must be applied to the top block to make it slide across the bottom block.

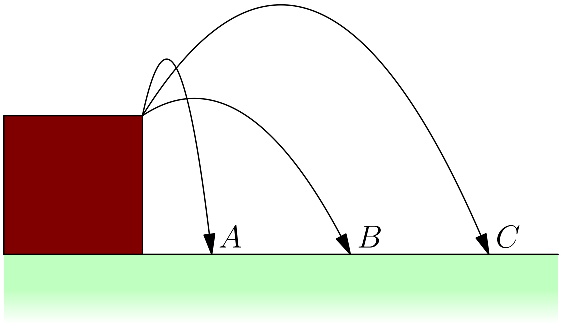

Figure 3.4: Three projectiles thrown off a cliff.

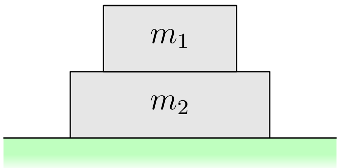

Figure 3.5: Two stacked blocks.

Problem 3E. Two blocks are connected by a string, and one block is on a platform as shown in Figure 3.6. The coefficient of kinetic friction between the platform and the block is μ_k. Assuming the system is moving, determine the tension in the string and the magnitude of the accelerations of the blocks.

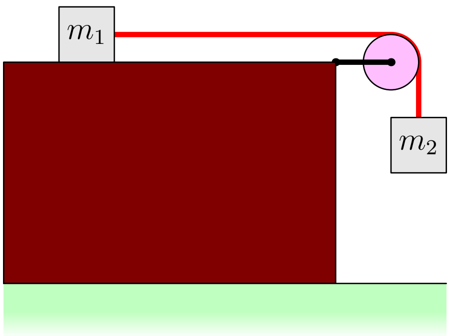

Figure 3.6: A half-Atwood machine.

Problem 3F. A packing crate with mass 115 kg is slid up a 5.00 m long ramp which makes an angle of 20.0° with respect to the horizontal by an applied force of 1.00×10^3 N directed parallel to the ramp's incline. A frictional force of magnitude 4.00×10^2 N resists the motion. If the crate starts from rest, what is its speed at the top of the ramp?

Problem 3G. Two masses are attached with pulleys by a massless rope on an inclined plane as shown in Figure 3.7. All surfaces are frictionless. If the masses are released from rest, then characterize the motion of the inclined plane.

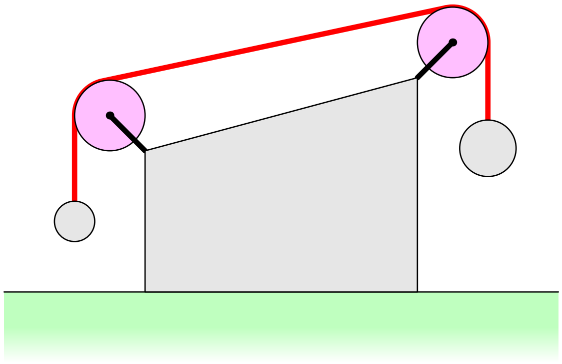

Figure 3.7: Two masses on a frictionless inclined plane.

Problem 3H. A block on a ramp is given an initial speed upwards along the ramp, as shown in Figure 3.8. It slides upward for a time t_u, traveling some distance, and then slides downward for a time t_d until it returns to its original position. If the height of the incline is 0.6 times its diagonal length and the coefficient of kinetic friction between the block and the incline is 0.5, what is $\frac{t_d}{t_u}$?

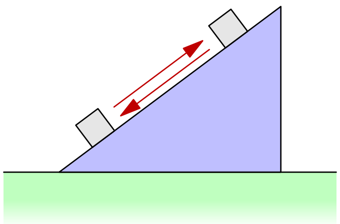

Figure 3.8: A block sliding up and down a ramp.

Problem 3I. A block is released from rest at the top of a fixed, frictionless ramp with horizontal length 1 m and inclination θ, as shown in Figure 3.9. What value of θ minimizes the time needed for the block to reach the bottom of the ramp?

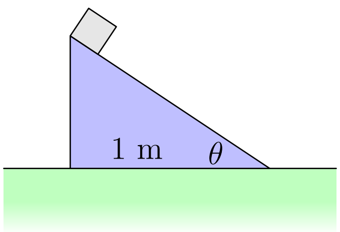

Figure 3.9: A block on a ramp.

Problem 3J. A cannon that ejects projectiles at a fixed speed is situated at the base of an ramp with angle θ, as shown in Figure 3.10. In terms of θ, at what angle should the cannon be positioned so that the projectile will reach the farthest up the ramp? Assume the ramp is sufficiently large.

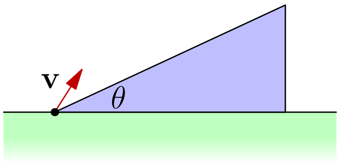

Figure 3.10: A projectile fired up a ramp.

Problem 3K. A point mass m sits on a long block, also of mass m, which rests on the floor. The coefficient of static and kinetic friction between the mass and the block is μ, and the coefficient of static and kinetic friction between the block and the floor is $\frac{1}{3}\mu$. An impulse gives a horizontal momentum with magnitude p to the point mass. After a long time, how far has the point mass moved relative to the block? Assume the mass does not fall off the block.

Problem 3L. An object starts at rest and starts falling. In terms of Earth's gravity g, the density of air ρ, the object's reference area A, and the drag coefficient C_D, determine the object's velocity as a function of time, without ignoring air resistance.

4 Rotational dynamics

Rotational dynamics studies the rotating component of an object's motion. In this chapter, all objects are modeled as rigid bodies whose point particles don't all lie on a common line. Certain quantities are measured with respect to the origin of a reference frame or from an axis ℓ.

§4.1 Moment of inertia

Prototypical example for this section: a spinning hard drive.

The **moment of inertia** of a point particle with respect to an axis ℓ is

$$I_\ell = mr^2,$$

where m is the mass of the particle and r is the distance from the particle to ℓ. The **moment of inertia** of a rigid body with respect to ℓ is the sum of the moment of inertiae of each particle in the rigid body with respect to ℓ; that is,

$$I_\ell = \sum_i m_i r_i^2.$$

Moment of inertia is the rotational analogue of mass and its units are $\mathrm{kg} \cdot \mathrm{m}^2$. It can be thought of as a measure of the rigid body's resistance to rotation around ℓ, in the same way mass is a resistance to linear acceleration.

The moment of inertiae of solid objects can be found through an integral. The moment of inertiae of some common objects are below.

Corollary 4.1.1 (Rod moment of inertia)
The moment of inertia of an infinitely thin rod with length l and mass m about a line through its center perpendicular to the rod is

$$I = \frac{1}{12} ml^2.$$

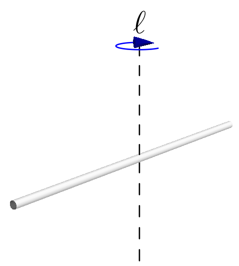

Figure 4.1: A rod with perpendicular bisector ℓ.

Proof. By interpreting each segment on the rod with length dx as a point particle with mass $m\frac{dx}{l}$, the rod's moment of inertia is the infinite sum

$$\int_{-\frac{l}{2}}^{-\frac{l}{2}} \left(\frac{m}{l}\right) x^2\, dx = \frac{m}{3l}\left(\frac{l}{2}\right)^3 - \frac{m}{3l}\left(-\frac{l}{2}\right)^3 = \frac{1}{12}ml^2. \qquad \square$$

Corollary 4.1.2 (Solid cylinder moment of inertia)
The moment of inertia of a solid uniform cylinder with radius r and mass m about its axis of symmetry is

$$I = \frac{1}{2}mr^2.$$

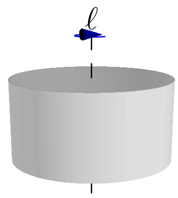

Figure 4.2: A cylinder with axis of symmetry ℓ.

Proof. Note that the length of the cylinder is irrelevant, so assume the cylinder is infinitely thin. By interpreting each infinitesimal rectangle located at the polar coordinates (s, θ) as a point particle with mass $\frac{m}{\pi r^2}(d\theta)(s\,ds)$, the cylinder's moment of inertia is the infinite sum

$$I = \int_0^{2\pi}\int_0^r \left(\frac{ms}{\pi r^2}\right) s^2\, ds\, d\theta = \int_0^{2\pi} \frac{m}{\pi r^2}\cdot\frac{1}{4}r^4\, d\theta = 2\pi\cdot\frac{m}{\pi r^2}\cdot\frac{1}{4}r^4 = \frac{1}{2}mr^2. \qquad \square$$

Corollary 4.1.3 (Hollow sphere moment of inertia)
The moment of inertia of an infinitely thin uniform spherical shell with radius r and mass m about a line through its center is

$$I = \frac{2}{3}mr^2.$$

Proof. By interpreting each infinitesimal rectangle on the surface of the sphere at the spherical coordinates (r, θ, φ) as a point particle with mass $\frac{m}{4\pi r^2}(r\,d\varphi)(r\sin\varphi\, d\theta)$, the shell's moment of inertia is the infinite sum

$$I = \int_0^{\pi}\int_0^{2\pi}\left(\frac{m}{4\pi r^2}\cdot r^2\sin\varphi\right)\cdot(r\sin\varphi)^2\, d\theta\, d\varphi = \int_0^{\pi}\frac{mr^2(\sin\varphi)^3}{2}d\varphi = \frac{2}{3}mr^2. \qquad \square$$

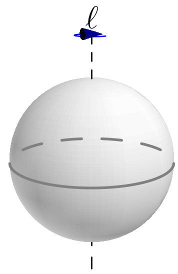

Figure 4.3: A shell with axis of symmetry ℓ.

Corollary 4.1.4 (Solid sphere moment of inertia)

The moment of inertia of a solid uniform sphere with radius r and mass m about a line through its center is

$$I = \frac{2}{5}mr^2.$$

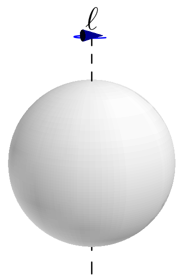

Figure 4.4: A sphere with axis of symmetry ℓ.

Proof. Interpret the solid sphere as a collection of shells with radius s, infinitesimal thickness ds, and mass $\frac{m}{\frac{4}{3}\pi r^3} \cdot 4\pi s^2 ds$. Since the moment of inertia of a shell with radius s is $\frac{2}{3}ms^2$, the moment of inertia of the sphere is

$$I = \int_0^r \frac{2}{3}\left(\frac{m}{\frac{4}{3}\pi r^3} \cdot 4\pi s^2\right) s^2 ds = \int_0^r \frac{2s^4}{r^3} ds = \frac{2}{5}mr^2. \qquad \square$$

The **stretch rule** and the **perpendicular axis theorem** can be used to find the moments of inertia for a variety of rigid bodies and axes.

Theorem 4.1.5 (Stretch rule)

The moment of inertia of a rigid body around a line ℓ is unchanged when the object is stretched in a direction parallel to ℓ.

Proof. The distance of each point particle to ℓ remains unchanged when the object is stretched in a direction parallel to ℓ, so the moment of inertia remains unchanged as well. $\square$

Theorem 4.1.6 (Perpendicular axis theorem)

Suppose a rigid body is contained entirely within a plane $\mathcal{P}$, and let ℓ be a line perpendicular to $\mathcal{P}$. Then the moment of inertia of the rigid body about ℓ equals the sum of the moments of inertia of rigid body about any two perpendicular lines ℓ_x and ℓ_y through $\mathcal{P}$ passing through ℓ.

Proof. Assume $\mathcal{P}$ is the xy-plane, and work in a reference frame where ℓ_x is the x-axis, ℓ_y is the y-axis, and ℓ_z is the z-axis. Then by the Pythagorean theorem,

$$\begin{aligned} I_\ell &= \sum_i m_i \left\|\mathbf{x}_i\right\|^2 \\ &= \sum_i m_i \left(\left\|\text{proj}_{\hat{\mathbf{i}}}(\mathbf{x}_i)\right\|^2 + \left\|\text{proj}_{\hat{\mathbf{j}}}(\mathbf{x}_i)\right\|^2\right) \\ &= \sum_i \left[m_i \left\|\text{proj}_{\hat{\mathbf{i}}}(\mathbf{x}_i)\right\|^2\right] + \sum_i \left[\left\|\text{proj}_{\hat{\mathbf{j}}}(\mathbf{x}_i)\right\|^2\right] \\ &= I_{\ell_x} + I_{\ell_y}. \end{aligned}$$

□

§4.2 Moment of inertia tensor

The **moment of inertia tensor** of a rigid body is a 3×3 matrix that can be used to compute the moment of inertia of a rigid body with respect to any axis through the origin of the reference frame. It is defined as

$$\mathbf{I} = \sum_i m_i [\mathbf{x}_i]_\times^\top [\mathbf{x}_i]_\times = -\sum_i m_i [\mathbf{x}_i]_\times^2.$$

Using the definition of the matrix cross product operator, the moment of inertia tensor is equivalently

$$\begin{aligned} \mathbf{I} &= -\sum_i m_i \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}^2 \\ &= -\sum_i m_i \begin{bmatrix} -\left(y_i^2 + z_i^2\right) & x_i y_i & x_i z_i \\ x_i y_i & -\left(x_i^2 + z_i^2\right) & y_i z_i \\ x_i z_i & y_i z_i & -(x_i^2 + y_i^2) \end{bmatrix} \\ &= \begin{bmatrix} \sum_i m_i (y_i^2 + z_i^2) & -\sum_i x_i y_i & -\sum_i x_i z_i \\ -\sum_i x_i y_i & \sum_i m_i (x_i^2 + z_i^2) & -\sum_i y_i z_i \\ -\sum_i x_i z_i & -\sum_i y_i z_i & \sum_i m_i (x_i^2 + y_i^2) \end{bmatrix}, \end{aligned}$$

where $\mathbf{x}_i = x_i \hat{\mathbf{i}} + y_i \hat{\mathbf{j}} + z_i \hat{\mathbf{k}}$. Note that the moment of inertia tensor is Hermitian; that is, symmetric.

Remark 4.2.1 — The moment of inertia tensor is dependent on the frame of reference, so it generally changes as an object moves or rotates.

The following corollary shows how the moment of inertia tensor can be used to calculate the moment of inertia of any line through the origin.

Corollary 4.2.2 (Moment of inertia tensor encodes all moments of inertia)
The moment of inertia of a rigid body with respect to a line ℓ through the origin is

$$I_\ell = \hat{\mathbf{n}} \cdot \mathbf{I}\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector parallel to ℓ.

Proof. Let θ_i be the angle between $\mathbf{x}_i$ and $\hat{\mathbf{n}}$. Then

$$\begin{aligned}
\hat{\mathbf{n}} \cdot \mathbf{I}\hat{\mathbf{n}} &= \hat{\mathbf{n}} \cdot \sum_i -m_i[\mathbf{x}_i]_\times^2 \hat{\mathbf{n}} \\
&= \hat{\mathbf{n}} \cdot \sum_i -m_i \mathbf{x}_i \times (\mathbf{x}_i \times \hat{\mathbf{n}}) \\
&= \sum_i m_i \hat{\mathbf{n}} \cdot ((\mathbf{x}_i \cdot \mathbf{x}_i)\hat{\mathbf{n}} - (\mathbf{x}_i \cdot \hat{\mathbf{n}})\mathbf{x}_i) \\
&= \sum_i m_i \left((\mathbf{x}_i \cdot \mathbf{x}_i) - (\mathbf{x}_i \cdot \hat{\mathbf{n}})^2\right) \\
&= \sum_i m_i \left(\|\mathbf{x}_i\|^2 - (\|\mathbf{x}_i\| \cos\theta_i)^2\right) \\
&= \sum_i m_i \left(\|\mathbf{x}_i\| \sin\theta_i\right)^2 \\
&= \sum_i m_i r_i^2 \\
&= I_\ell.
\end{aligned}$$

□

Corollary 4.2.3 (Moment of inertia tensor is invertible)
The moment of inertia tensor is invertible.

Proof. Suppose for contradiction that $\mathbf{I}\mathbf{v} = \mathbf{0}$ for some nonzero vector $\mathbf{v}$. Let $\hat{\mathbf{n}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ and let ℓ be a line through the origin parallel to $\hat{\mathbf{n}}$. By Corollary 4.2.2,

$$I_\ell = \hat{\mathbf{n}} \cdot \mathbf{I}\hat{\mathbf{n}} = 0,$$

a contradiction since I_ℓ is a sum of squared distances to ℓ, which cannot all be zero. □

§4.3 Parallel axis theorem

The **parallel axis theorem** can be used to compute the moment of inertia tensor, given its moment of inertia tensor with respect to its center of mass.

Theorem 4.3.1 (Parallel axis theorem, tensor form)
The moment of inertia tensor of a rigid body with mass m and center of mass $\mathbf{x}_{\text{cm}}$ is

$$\mathbf{I} = \mathbf{I}_{\text{cm}} + m\,[\mathbf{x}_{\text{cm}}]_\times^\top [\mathbf{x}_{\text{cm}}]_\times = \mathbf{I}_{\text{cm}} - m\,[\mathbf{x}_{\text{cm}}]_\times^2\,,$$

where $\mathbf{I}_{\text{cm}}$ is the moment of inertia tensor of the rigid body in the reference frame with the same orientation centered at $\mathbf{x}_{\text{cm}}$.

Proof. Let $\mathbf{x}_i$ denote the position of each point particle in the rigid body relative to its center of mass. Then

$$\begin{aligned}
\mathbf{I} - \mathbf{I}_{\text{cm}} &= -\sum_i m_i \left([\mathbf{x}_i + \mathbf{x}_{\text{cm}}]_\times^2 - ([\mathbf{x}_i]_\times)^2 \right) \\
&= -\sum_i m_i \left([\mathbf{x}_i]_\times [\mathbf{x}_{\text{cm}}]_\times + [\mathbf{x}_{\text{cm}}]_\times [\mathbf{x}_i]_\times - [\mathbf{x}_{\text{cm}}]_\times^2 \right) \\
&= \left(\sum_i m_i [\mathbf{x}_{\text{cm}}]_\times^2 \right) - \left(\sum_i m_i [\mathbf{x}_i]_\times \right) [\mathbf{x}_{\text{cm}}]_\times - [\mathbf{x}_{\text{cm}}]_\times \left(\sum_i m_i [\mathbf{x}_i]_\times \right) \\
&= -\sum_i m_i [\mathbf{x}_{\text{cm}}]_\times^2,
\end{aligned}$$

where the last equality follows because

$$\sum_i m_i [\mathbf{x}_i]_\times$$

is the zero matrix by considering each entry in the matrix separately. □

The statement of the parallel axis theorem more commonly found in an introductory physics course is the following.

Corollary 4.3.2 (Parallel axis theorem)
Let ℓ_{cm} be the line through the center of mass of the rigid body parallel to ℓ. Then

$$I_\ell = I_{\ell_{\text{cm}}} + md^2,$$

where d is the distance between ℓ and ℓ_{cm}.

Proof. Let $\hat{\mathbf{n}}$ be a unit vector parallel to ℓ. By taking the statement of the parallel axis theorem in tensor form, multiplying the right side by $\hat{\mathbf{n}}$, and dotting the left side with $\hat{\mathbf{n}}$, one obtains

$$\begin{aligned}
I_\ell &= I_{\ell_{\text{cm}}} - m\hat{\mathbf{n}} \cdot [\mathbf{x}_{\text{cm}}]_\times^2 \hat{\mathbf{n}} \\
&= I_{\ell_{\text{cm}}} - m\hat{\mathbf{n}} \cdot \mathbf{x}_{\text{cm}} \times (\mathbf{x}_{\text{cm}} \times \hat{\mathbf{n}}) \\
&= I_{\ell_{\text{cm}}} + m\hat{\mathbf{n}} \cdot ((\mathbf{x}_{\text{cm}} \cdot \mathbf{x}_{\text{cm}})\hat{\mathbf{n}} - (\mathbf{x}_{\text{cm}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}) \\
&= I_{\ell_{\text{cm}}} + m \left(\|\mathbf{x}_{\text{cm}}\|^2 - (\|\mathbf{x}_{\text{cm}}\| \sin\theta)^2 \right) \\
&= I_{\ell_{\text{cm}}} + m (\|\mathbf{x}_{\text{cm}}\| \cos\theta)^2 \\
&= I_{\ell_{\text{cm}}} + md^2,
\end{aligned}$$

where θ is the angle between $\mathbf{r}_{\text{cm}}$ and $\hat{\mathbf{n}}$. □

§4.4 Torque

Prototypical example for this section: sitting on a seesaw.

The **torque** of a force applied to an object relative to $\mathbf{x}_0$ – that is, measured from $\mathbf{x}_0$ – is the cross product of its position vector relative to $\mathbf{x}_0$ and the force vector. In other words,

$$\boldsymbol{\tau} = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{F} = \mathbf{x}' \times \mathbf{F}.$$

The **net torque** exerted on an object relative to $\mathbf{x}_0$ is the sum of all torques applied to the object relative to $\mathbf{x}_0$. Torque is measured in **newton-meters**, denoted $\mathrm{N}\cdot\mathrm{m} = \frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2}$, and it can be thought of as the rotational analogue of force. By properties of the cross product, the torque vector is always perpendicular to the force and position vectors, and

$$\|\boldsymbol{\tau}\| = \left\|\mathbf{x}'\right\| \|\mathbf{F}\| \sin\theta,$$

where θ is the angle between $\mathbf{x}'$ and $\mathbf{F}$.

Remark 4.4.1 — When an object is fixed so that it rotates around a fixed point, it is useful to consider the torque measured from the fixed point. It is also useful to consider the net torque measured from an object's center of mass.

Corollary 4.4.2 (Torque from gravity)
The magnitude of the net torque exerted on an rigid body with mass m by gravity relative to x_0 is

$$\|\boldsymbol{\tau}\| = \|\mathbf{x}_{\mathrm{cm}} - \mathbf{x}_0\| \left\|-mg\hat{\mathbf{k}}\right\| \sin\theta = \left\|\mathbf{x}'_{\mathrm{cm}}\right\| mg\sin\theta,$$

where θ is the angle between $\mathbf{x}'_{\mathrm{cm}}$ and $\hat{\mathbf{k}}$.

Proof. Since the planet (or other celestial body) exerting the gravitational force is sufficiently far away from the rigid body, the gravitational force exerted on every particle in the rigid body all point in the approximately the same direction $\hat{\mathbf{k}}$ with a magnitude approximately proportional to the mass of the particle. The net torque exerted on the rigid body relative to $\mathbf{x}_0$ is then

$$\begin{aligned}
\boldsymbol{\tau} &= \sum_i (\mathbf{x}_i - \mathbf{x}_0) \times \left(-m_i g\hat{\mathbf{k}}\right) \\
&= \left(\sum_i m_i(\mathbf{x}_i - \mathbf{x}_0)\right) \times -g\hat{\mathbf{k}} \\
&= \left(\sum_i m_i(\mathbf{x}_i - \mathbf{x}_{\mathrm{cm}})\right) \times -g\hat{\mathbf{k}} + \left(\sum_i m_i(\mathbf{x}_{\mathrm{cm}} - \mathbf{x}_0)\right) \times -g\hat{\mathbf{k}} \\
&= \mathbf{0} + m\mathbf{x}' \times -g\hat{\mathbf{k}}
\end{aligned}$$

Thus

$$\|\boldsymbol{\tau}\| = \left\|\mathbf{x}'_{\mathrm{cm}}\right\| mg\sin\theta. \qquad \square$$

Remark 4.4.3 — In particular, gravity exerts no torque on an object relative to its center of mass.

Exercise 4.4.4. Two children with masses m_1 and m_2 are sitting on opposite ends of a horizontal seesaw. If they are sitting at a distance of d_1 and d_2 away from the pivot of the seesaw, respectively, then determine the torque each child exerts on the seesaw relative to the pivot point.

§4.5 Angular momentum

Prototypical example for this section: the Earth relative to the Sun.

The **orbital angular momentum** of a point particle relative to $\mathbf{x}_0$ is a vector equal to the cross product of its position vector relative to $\mathbf{x}_0$ and its momentum relative to $\mathbf{x}_0$; that is,

$$\mathbf{L} = (\mathbf{x} - \mathbf{x}_0) \times (m(\mathbf{v} - \mathbf{v}_0)) = \mathbf{x}' \times \mathbf{p}',$$

where $\mathbf{p}'$ is the momentum relative to $\mathbf{x}_0$. The **angular momentum** of a system, such as a rigid body, relative to $\mathbf{x}_0$ is a vector equal to the sum of the orbital angular momenta of the point particles relative to $\mathbf{x}_0$ that comprise the system; that is,

$$\mathbf{L} = \sum_i (\mathbf{x}_i - \mathbf{x}_0) \times (m_i(\mathbf{v}_i - \mathbf{v}_0)) = \sum_i \mathbf{x}'_i \times \mathbf{p}'_i.$$

The **spin angular momentum** of a system is equal to its orbital angular momentum relative to its center of mass; that is,

$$\mathbf{L}_s = \sum_i (\mathbf{x}_i - \mathbf{x}_{\text{cm}}) \times (m_i(\mathbf{v}_i - \mathbf{v}_{\text{cm}})) = \sum_i (\mathbf{x}_i - \mathbf{x}_{\text{cm}}) \times (\mathbf{p}_i - m_i\mathbf{v}_{\text{cm}}).$$

Note that spin angular momentum does not depend on a reference point.

Orbital angular momentum, angular momentum, and spin angular momentum are all measured in $\frac{\text{kg}\cdot\text{m}^2}{\text{s}}$. The following result exemplifies the necessity for each type of angular momentum.

Corollary 4.5.1 (Angular momentum decomposition)

The angular momentum of a system relative to $\mathbf{x}_0$ is the the sum of its spin angular momentum and the orbital angular momentum of its center of mass relative to $\mathbf{x}_0$; that is,

$$\mathbf{L} = \mathbf{L}_s + \mathbf{L}_{\text{cm}}.$$

Proof. Let $\mathbf{x}'_{\text{cm}} = \mathbf{x}_{\text{cm}} - \mathbf{x}_0$ be the center of mass relative to $\mathbf{x}_0$. By expansion,

$$\begin{aligned}
\sum_i \mathbf{x}'_i \times \mathbf{p}_i &= \sum_i \left[\mathbf{x}'_{\text{cm}} \times \mathbf{p}_i\right] + \sum_i \left[(\mathbf{x}'_i - \mathbf{x}'_{\text{cm}}) \times \mathbf{p}_i\right] \\
&= \mathbf{x}'_{\text{cm}} \times \mathbf{p} + \sum_i \left[(\mathbf{x}'_i - \mathbf{x}'_{\text{cm}}) \times (\mathbf{p}_i - m_i\mathbf{v}_{\text{cm}})\right] + \sum_i \left[(\mathbf{x}'_i - \mathbf{x}'_{\text{cm}}) \times m_i\mathbf{v}_{\text{cm}}\right] \\
&= \mathbf{L}_{\text{cm}} + \mathbf{L}_s + \mathbf{v}_{\text{cm}} \times \sum_i \left[m_i(\mathbf{x}'_i - \mathbf{x}'_{\text{cm}})\right] \\
&= \mathbf{L}_{\text{cm}} + \mathbf{L}_s.
\end{aligned}$$

□

The following result makes it clear why angular momentum is the rotational analogue of momentum.

Theorem 4.5.2 (Angular momentum is moment of inertia tensor times angular velocity)

Let $\mathbf{x}_0$ be any fixed reference point. The angular momentum of a system with respect to $\mathbf{x}_0$ is the product of its moment of inertia tensor with respect to $\mathbf{x}_0$ and its angular velocity; that is,

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}.$$

Proof. By expansion,

$$\mathbf{I}\boldsymbol{\omega} = -\sum_i m_i [\mathbf{x}_i']_\times^2 \boldsymbol{\omega} = -\sum_i m_i \mathbf{x}_i' \times (\mathbf{x}_i' \times \boldsymbol{\omega}) = \sum_i \mathbf{x}_i' \times m_i \mathbf{v}_i' = \mathbf{L}. \qquad \square$$

§4.6 Euler's second law

Euler's second law equates the net torque on an object with the rate of change of its momentum.

Theorem 4.6.1 (Euler's second law of motion)
In an inertial reference frame, the rate of change in angular momentum of a system about a *fixed* reference $\mathbf{x}_0$ is equal to the net torque acting on the system about $\mathbf{x}_0$; that is,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

Additionally, the rate of change in spin angular momentum of a system is equal to the net torque acting on the system about its center of mass; that is,

$$\boldsymbol{\tau}_{\mathrm{cm}} = \frac{d\mathbf{L}_s}{dt}.$$

Proof. Let $\mathbf{F}_i$ be the (possibly zero) external force acting on each point particle i in the system, and let particle i exert a force $\mathbf{F}_{ij}$ on particle j. Then the time derivative of its angular momentum relative to $\mathbf{x}_0$ is

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \sum_i \frac{d}{dt}\left[\mathbf{x}_i' \times m_i(\mathbf{v}_i')\right] \\
&= \sum_i \mathbf{v}_i' \times m_i(\mathbf{v}_i') + \mathbf{x}_i' \times \frac{d}{dt} m_i(\mathbf{v}_i) \\
&= \sum_i \mathbf{x}_i' \times \frac{d}{dt} m_i(\mathbf{v}_i') \\
&= \sum_i \left[\mathbf{x}_i' \times \left(\mathbf{F}_i + \sum_{j\neq i} \mathbf{F}_{ji}\right)\right] \\
&= \sum_i \left[\mathbf{x}_i' \times \mathbf{F}_i\right] + \sum_{i<j} \left[\mathbf{x}_i' \times \mathbf{F}_{ji} + \mathbf{x}_j' \times \mathbf{F}_{ij}\right] \\
&= \boldsymbol{\tau} + \sum_{i<j} \left[(\mathbf{x}_i' - \mathbf{x}_j') \times \mathbf{F}_{ji}\right] \\
&= \boldsymbol{\tau},
\end{aligned}$$

where the last equality follows because $\mathbf{x}_i - \mathbf{x}_j' = \mathbf{x}_i - \mathbf{x}_j$ is parallel to $\mathbf{F}_{ji}$.

Similarly, the time derivative of its angular momentum relative to $\mathbf{x}_{\mathrm{cm}}$ is

$$\begin{aligned}
\frac{d\mathbf{L}_s}{dt} &= \sum_i \frac{d}{dt}\left[(\mathbf{x}_i - \mathbf{x}_{\mathrm{cm}}) \times m_i(\mathbf{v}_i - \mathbf{v}_{\mathrm{cm}})\right] \\
&= \sum_i (\mathbf{v}_i - \mathbf{v}_{\mathrm{cm}}) \times m_i(\mathbf{v}_i - \mathbf{v}_{\mathrm{cm}}) + (\mathbf{x}_i - \mathbf{x}_{\mathrm{cm}}) \times \frac{d}{dt} m_i(\mathbf{v}_i - \mathbf{v}_{\mathrm{cm}})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i (\mathbf{x}_i - \mathbf{x}_{\text{cm}}) \times \frac{d}{dt} m_i (\mathbf{v}_i - \mathbf{v}_{\text{cm}}) \\
&= \sum_i \left[(\mathbf{x}_i - \mathbf{x}_{\text{cm}}) \times \frac{d}{dt} m_i \mathbf{v}_i \right] - \sum_i \left[(\mathbf{x}_i - \mathbf{x}_{\text{cm}}) \times \frac{d}{dt} m_i \mathbf{v}_{\text{cm}} \right] \\
&= \sum_i \left[(\mathbf{x}_i - \mathbf{x}_{\text{cm}}) \times \left(\mathbf{F}_i + \sum_{j \neq i} \mathbf{F}_{ji} \right) \right] - \sum_i [m_i (\mathbf{x}_i - \mathbf{x}_{\text{cm}})] \times \mathbf{a}_{\text{cm}} \\
&= \sum_i [(\mathbf{x}_i - \mathbf{x}_{\text{cm}}) \times \mathbf{F}_i] + \sum_{i<j} [\mathbf{x}_i \times \mathbf{F}_{ji} + \mathbf{x}_j \times \mathbf{F}_{ij}] - \mathbf{0} \\
&= \boldsymbol{\tau}_{\text{cm}} + \sum_{i<j} [(\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{F}_{ji}] \\
&= \boldsymbol{\tau}_{\text{cm}}.
\end{aligned}
$$

□

Unlike linear dynamics, torque is generally not the product of the moment of inertia tensor and angular acceleration; this is only true under special conditions. However, a rotational equivalent of $\mathbf{F} = m\mathbf{a}$ does exist, called **Euler's rotation equation**.

Theorem 4.6.2 (Euler's rotation equation)
The net torque exerted on a rigid body is

$$\boldsymbol{\tau} = \mathbf{I}_{\text{cm}} \boldsymbol{\alpha} + \boldsymbol{\omega} \times \mathbf{L}_s.$$

Proof. Let $\mathbf{I}_{\text{body}}$ be the moment of inertia tensor of a rigid body in a reference frame fixing the rigid body centered at its center of mass, so that

$$\mathbf{I}_{\text{cm}} = \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{-1} = \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\top}$$

for some rotation matrix $\mathbf{R}$, which varies as time passes. Note that

$$\frac{d\mathbf{R}}{dt} = [\boldsymbol{\omega}]_\times \mathbf{R},$$

since the columns of $\mathbf{R}$ have constant length – in particular, unit length. Differentiating angular momentum gives

$$
\begin{aligned}
\boldsymbol{\tau} = \frac{d\mathbf{L}_s}{dt} &= \frac{d}{dt} \left[\mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\top}\boldsymbol{\omega} \right] \\
&= \frac{d\mathbf{R}}{dt} \mathbf{I}_{\text{body}}\mathbf{R}^{\top}\boldsymbol{\omega} + \mathbf{R}\mathbf{I}_{\text{body}} \left(\frac{d\mathbf{R}}{dt} \right)^{\top} \boldsymbol{\omega} + \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\top} \frac{d\boldsymbol{\omega}}{dt} \\
&= [\boldsymbol{\omega}_\times] \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\top}\boldsymbol{\omega} + \mathbf{R}\mathbf{I}_{\text{body}} \left([\boldsymbol{\omega}]_\times \mathbf{R} \right)^{\top} \boldsymbol{\omega} + \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\top}\boldsymbol{\alpha} \\
&= [\boldsymbol{\omega}]_\times \mathbf{I}_{\text{cm}}\boldsymbol{\omega} + \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\top} [\boldsymbol{\omega}]_\times^{\top} \boldsymbol{\omega} + \mathbf{I}_{\text{cm}}\boldsymbol{\alpha} \\
&= [\boldsymbol{\omega}]_\times \mathbf{L}_s + \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\top} \left(-[\boldsymbol{\omega}]_\times \right) \boldsymbol{\omega} + \mathbf{I}_{\text{cm}}\boldsymbol{\alpha} \\
&= \boldsymbol{\omega} \times \mathbf{L}_s + \mathbf{I}_{\text{cm}}\boldsymbol{\alpha},
\end{aligned}
$$

where the first line follows from Euler's second law, the second line follows from the product rule, the fifth line follows because $[\boldsymbol{\omega}]_\times$ is skew-symmetric, and the sixth line follows because $\boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$. □

Much like the law of conservation of momentum, the **law of conservation of angular momentum** holds in a system not acted upon by external torques. There are two parts of the law of conservation of angular momentum: one for angular momentum relative to $\mathbf{x}_0$ and one for spin angular momentum.

Theorem 4.6.3 (Law of conservation of angular momentum)

In a system not acted upon by external torques relative to a fixed point $\mathbf{x}_0$, the total angular momenta relative to $\mathbf{x}_0$ is constant. Additionally, when not acted upon by external torques relative to the center of mass of the system, the spin angular momentum of a system is constant.

Proof. Both results follow from Euler's second law, since

$$\boldsymbol{\tau} = \mathbf{0} = \frac{d\mathbf{L}}{dt}$$

for the first case, and

$$\boldsymbol{\tau} = \mathbf{0} = \frac{d\mathbf{L}_s}{dt}$$

for the second case. □

Exercise 4.6.4. Suppose a bullet is shot upwards and gets lodged in a block of wood near its edge, sending the block and the bullet inside it spinning. Before the collision, no objects were spinning, but after the collision, both objects spin. Explain why this does not violate conservation of angular momentum.

§4.7 Fixed-axis rotation

Although three-dimensional rigid body motion is in general complex, many of the results above simplify nicely when considering fixed-axis rotation.

Theorem 4.7.1 (Angular momentum in fixed-axis rotation)

Suppose a rigid body is rotating around a fixed axis ℓ. Then the component of angular momentum in the direction of angular velocity is the product of its moment of inertia around ℓ and its angular velocity; that is,

$$\operatorname{proj}_{\boldsymbol{\omega}}(\mathbf{L}) = I_\ell \boldsymbol{\omega}.$$

Proof. Using the definitions,

$$\begin{aligned}
\operatorname{proj}_{\boldsymbol{\omega}}(\mathbf{L}) &= \sum_i \operatorname{proj}_{\boldsymbol{\omega}}(\mathbf{r} \times m\mathbf{v}) \\
&= \sum_i \mathbf{r}' \times m\mathbf{v} \\
&= \sum_i \mathbf{r}' \times (m\boldsymbol{\omega} \times \mathbf{r}') \\
&= \sum_i mr^2 \boldsymbol{\omega} \\
&= I_\ell \boldsymbol{\omega},
\end{aligned}$$

where $\mathbf{r}_i'$ is the component of $\mathbf{r}_i$ parallel to the plane perpendicular to $\boldsymbol{\omega}$. □

Corollary 4.7.2 (Torque in fixed-axis rotation)
Suppose a rigid body is rotating around a fixed axis ℓ. Then the component of torque in the direction of angular velocity is the product of its moment of inertia around ℓ and its angular acceleration; that is,

$$\operatorname{proj}_{\boldsymbol{\omega}}(\boldsymbol{\tau}) = I_\ell \boldsymbol{\alpha}.$$

Proof. This follows immediately from differentiating the previous result and citing Euler's second law. □

§4.8 Principal axes

Prototypical example for this section: the Earth's axis of rotation.

As one can deduce from Euler's rotation equation, it turns out that three-dimensional rigid body motion is complex and cannot be described simply as a fixed rotations. However, there exist special axes around which rigid bodies can rotate around at constant angular velocity, known as **principal axes**.

A principal axis of a rigid body is a line through its center of mass parallel to an eigenvector of its moment of inertia tensor relative to its center of mass $\mathbf{I}_{\text{cm}}$. As shown by the below results, a principal axis is, equivalently, an axis around which the rigid body can rotate around, torque-free, at constant angular velocity. Note that the principal axis of a rigid body relative to the rigid body does not depend on its orientation. The angular velocity vector and the spin angular momentum vector are parallel if and only if they point in the direction of a principal axis, since $\mathbf{L}_s = \mathbf{I}_{\text{cm}}\boldsymbol{\omega}$.

For objects rotating around a principal axis, a rotational analogue to Newton's second law holds.

Theorem 4.8.1 (Principal axis rotation)
If an object is rotating around an axis parallel to a principal axis, then

$$\boldsymbol{\tau} = \mathbf{I}_{\text{cm}}\boldsymbol{\alpha}.$$

Proof. This follows from Euler's rotation equation, since

$$\boldsymbol{\tau} = \mathbf{I}_{\text{cm}}\boldsymbol{\alpha} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \mathbf{I}_{\text{cm}}\boldsymbol{\alpha} + \boldsymbol{\omega} \times \lambda\boldsymbol{\omega} = \mathbf{I}_{\text{cm}}\boldsymbol{\alpha}$$

where λ is the corresponding eigenvalue of $\boldsymbol{\omega}$, as $\boldsymbol{\omega}$ is an eigenvector of $\mathbf{I}_{\text{cm}}$. □

> **Exercise 4.8.2.** Explain why a figure skater speeds up in rotation rate as she brings her arms in.

As a result, a rotational analogue of Newton's first law holds as well.

Corollary 4.8.3 (Torque-free principal axis rotation)
If no torques are acting on a rigid body and the rigid body is rotating around a principal axis, then its angular velocity is constant.

Proof. By Theorem 4.8.1,

$$\mathbf{I}_{\text{cm}}\boldsymbol{\alpha} = \boldsymbol{\tau} = \mathbf{0}.$$

Since $\mathbf{I}_{\text{cm}}$ is an invertible matrix, it must follow that $\boldsymbol{\alpha} = \mathbf{0}$, so $\boldsymbol{\omega}$ is constant. □

The **reflection theorem** and the **rotation theorem** give easy ways of finding principal axes of objects with some type of symmetry. The proof of both theorems relies on the fact that $\mathbf{I}$ is diagonalizable since it is symmetric; in particular, two eigenvectors of $\mathbf{I}$ with different eigenvalues must be perpendicular.

Theorem 4.8.4 (Reflection theorem)

If a rigid body has a plane of symmetry, then the line perpendicular to this plane through the center of mass is a principal axis.

Proof. Let $\mathbf{v}$ be an eigenvector of $\mathbf{I}_{\text{cm}}$ not in the plane of symmetry. By symmetry, its reflection over the plane $\mathbf{v}'$ must also be an eigenvector with the same eigenvalue. Hence, their difference $\mathbf{v} - \mathbf{v}'$, which is perpendicular to the plane of symmetry, is an eigenvector. □

Theorem 4.8.5 (Rotation theorem)

If a rigid body is rotationally symmetric about a line ℓ, then ℓ is a principal axis. Furthermore, if the degree of rotation is less than $180°$, then all lines through the center of mass perpendicular to ℓ are also principal axes.

Proof. Suppose the rigid body is rotationally symmetric with an angle of $\frac{360°}{n}$. Let $\mathbf{v}$ be an eigenvector of $\mathbf{I}_{\text{cm}}$ not perpendicular to ℓ. By symmetry, the n rotations of $\mathbf{v}$ around ℓ are also eigenvectors with the same eigenvalue, so the sum of all n vectors, which is a nonzero vector parallel to ℓ, is an eigenvector.

As a corollary, it follows that $\mathbf{I}_{\text{cm}}$ must have an eigenvector $\mathbf{u}$ perpendicular to ℓ as well, as $\mathbf{I}_{\text{cm}}$ is symmetric. If $n > 2$, then the n rotations of $\mathbf{u}$, which are all eigenvectors with the same eigenvalue, span the plane perpendicular to ℓ. Hence, all lines perpendicular to ℓ through the origin are principal axes if the degree of rotation is less than $180°$. □

If the diagonal elements of $\mathbf{I}_{\text{cm}}$'s diagonalization are all distinct – that is, there are three distinct eigenvalues – then there are exactly three different principal axes, each with distinct moments of inertia around them. While the rigid body could in theory rotate around all three without any external torques, the instability of certain rotations renders this impossible in reality; this result is known as the **tennis racket theorem**.

Theorem 4.8.6 (Tennis racket theorem)

Suppose a rigid body has three principal axes ℓ_1, ℓ_2, ℓ_3 with three distinct moments of inertia $I_1 > I_2 > I_3$. Then rotation around ℓ_1 and ℓ_3 is stable, but rotation around ℓ_2 is unstable. That is, any small perturbation in rotation around ℓ_1 or ℓ_3 will self-correct, but a perturbation in rotation around ℓ_2 will be amplified.

Proof. Let $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ be the unit vectors in pointing in the same direction as ℓ_1, ℓ_2, and ℓ_3; note that they form an orthonormal basis, so $\boldsymbol{\omega}$ can be written in the form

$$\boldsymbol{\omega} = \omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3$$

for some real numbers ω_1, ω_2, and ω_3. Substituting this into Euler's rotation equation and using $\mathbf{I}_i\mathbf{b}_i = I_i\mathbf{b}_i$ (which follows because $\mathbf{b}_i \cdot \mathbf{I}_i\mathbf{b}_i = I_i$) gives

$$\begin{aligned}
\boldsymbol{\tau} = \mathbf{0} = \mathbf{I}_{\text{cm}}\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} \\
&= \mathbf{I}_{\text{cm}}\left(\frac{d\omega_1}{dt}\mathbf{b}_1 + \frac{d\omega_2}{dt}\mathbf{b}_2 + \frac{d\omega_3}{dt}\mathbf{b}_3\right) \\
&\qquad + (\omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3) \times \mathbf{I}_{\text{cm}}(\omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3) \\
&= I_1\frac{d\omega_1}{dt}\mathbf{b}_1 + I_2\frac{d\omega_2}{dt}\mathbf{b}_2 + I_3\frac{d\omega_3}{dt}\mathbf{b}_3 \\
&\qquad + (\omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3) \times (I_1\omega_1\mathbf{b}_1 + I_2\omega_2\mathbf{b}_2 + I_3\omega_3\mathbf{b}_3) \\
&= \left(I_1\frac{d\omega_1}{dt} + I_3\omega_2\omega_3 - I_2\omega_3\omega_2\right)\mathbf{b}_1 \\
&\qquad + \left(I_2\frac{d\omega_2}{dt} + I_1\omega_3\omega_1 - I_3\omega_1\omega_3\right)\mathbf{b}_2 \\
&\qquad + \left(I_3\frac{d\omega_3}{dt} + I_2\omega_1\omega_2 - I_1\omega_2\omega_1\right)\mathbf{b}_3.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
I_1\frac{d\omega_1}{dt} &= (I_2 - I_3)\omega_2\omega_3 \\
I_2\frac{d\omega_2}{dt} &= (I_3 - I_1)\omega_3\omega_1 \\
I_3\frac{d\omega_3}{dt} &= (I_1 - I_2)\omega_1\omega_2.
\end{aligned}$$

Differentiating the first equation and substituting the other two gives

$$\begin{aligned}
I_1\frac{d^2\omega_1}{dt^2} &= (I_2 - I_3)\left(\frac{d\omega_2}{dt}\omega_3 + \omega_2\frac{d\omega_3}{dt}\right) \\
&= (I_2 - I_3)\left(\frac{I_3 - I_1}{I_2}\omega_3^2 + \frac{I_1 - I_2}{I_3}\omega_2^2\right)\omega_1
\end{aligned}$$

When rotating about ℓ_2, ω_2 is large while $\omega_3 \approx 0$, so $\frac{d^2\omega_1}{dt^2} = k\omega_1$ for some *positive* constant of proportionality k, since $(I_2 - I_3)(I_1 - I_2) > 0$. Hence any small perturbation in ω_1 will cause ω_1 to deviate from zero.

However, when rotating about ℓ_3, ω_3 is large while $\omega_2 \approx 0$, so $\frac{d^2\omega_1}{dt^2} = k\omega_1$ for some *negative* constant of proportionality k, since $(I_2 - I_3)(I_3 - I_1) < 0$. Hence any small perturbation in ω_1 will return to zero.

Similar analysis by differentiating the second and third equations and substituting gives

$$\begin{aligned}
I_2\frac{d^2\omega_2}{dt^2} &= (I_3 - I_1)\left(\frac{I_1 - I_2}{I_3}\omega_1^2 + \frac{I_2 - I_3}{I_1}\omega_3^2\right)\omega_2 \\
I_3\frac{d^2\omega_3}{dt^2} &= (I_1 - I_2)\left(\frac{I_2 - I_3}{I_1}\omega_2^2 + \frac{I_3 - I_1}{I_2}\omega_1^2\right)\omega_3.
\end{aligned}$$

One can check that ω_2 is stable when rotating around ℓ_1 and ℓ_3, and ω_3 is stable when rotating around ℓ_1 but not around ℓ_2. Combining each of these results shows that only rotation around ℓ_2 is unstable. $\square$

This effect is known as the tennis racket theorem because flipping a tennis racket around its intermediate axis – the one in the plane of the tennis racket perpendicular to the handle – almost always results in a corresponding half-turn; that is, the side of the tennis racket initially facing upwards before the flip will almost always face downwards after the flip. The effect also works for non-square rectangular objects, such as a cell phone.

§4.9 A few harder problems to think about

Problem 4A. Two massless rods are attached to frictionless pivots, with their ends touching. The distances between the pivot points and the endpoints of the rods are shown in Figure 4.5. Neglecting friction between the rods, if a force $\mathbf{F}$ is applied at the left end of the left rod, what force $\mathbf{F}'$ must be applied at the right end of the right rod to keep the system in equilibrium?

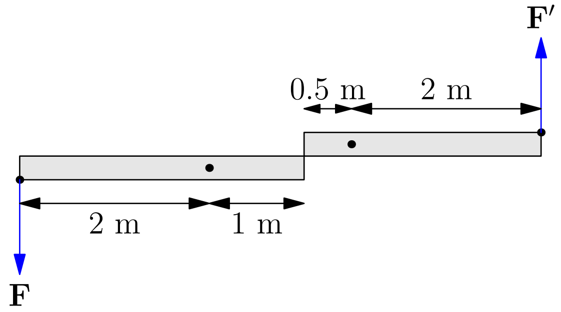

Figure 4.5: Two rods in equilibrium.

Problem 4B. A rectangular slab sits on a frictionless surface and a sphere sits on the slab, as shown in Figure 4.6. There is sufficient friction between the sphere and the slab such that the sphere will not slip relative to the slab. A force to the right is applied to the slab, with both the slab and the sphere initially at rest. Characterize the resulting motion of the sphere.

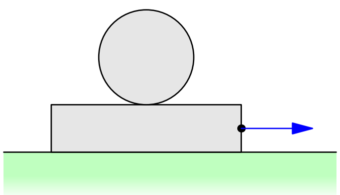

Figure 4.6: A sphere on a rectangular slab.

Problem 4C. A uniform stick of mass m is originally on a horizontal surface, as shown in Figure 4.7. One end is attached to a vertical string, which is pulled up with a constant tension force F_T so that the center of the mass of the stick moves upwards with acceleration a. Determine the normal force the ground exerts on the other end of the stick shortly after the right end of the stick leaves the ground.

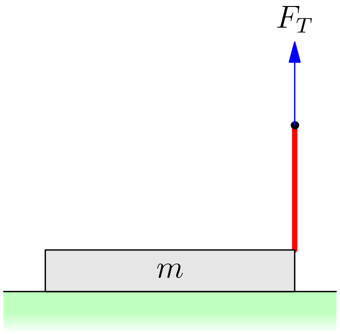

Figure 4.7: A stick pulled by a rope.

Problem 4D. A flat uniform disk of radius $2r$ has a hole of radius r removed from the center. The resulting annulus is then cut in half along a diameter. The remaining shape has mass m. Determine the moment of inertia of this shape about the axis of rotational symmetry of the original disk.

Problem 4E. An Atwood machine consists of two masses m_1 and m_2 hanging from a pulley with radius r and moment of inertia about its spinning axis I. If the string does not slip relative to the pulley, determine the magnitude of the acceleration of the blocks.

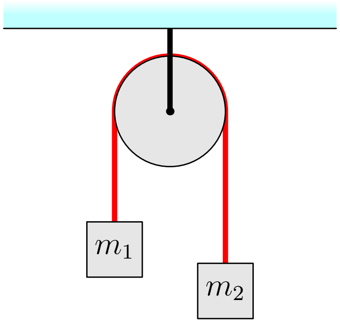

Figure 4.8: An Atwood machine.

Problem 4F. A toilet paper roll is dropped while holding its end so that it unrolls while dropping, and hits the ground in time t_1. If its end was not held fixed so that it did not unroll while dropping, it would have hit the ground in time t_2. Determine $\frac{t_1}{t_2}$.

Problem 4G. A solid uniform cylinder, a hollow uniform cylinder, a solid uniform sphere, and a hollow uniform sphere are rolling down an inclined plane from the same height without slipping. Rank the times in which they hit the ground.

Problem 4H. A uniform hollow spherical ball with mass m is placed on the ground with initial speed v_0 and zero spin angular velocity at time $t = 0$. The coefficient of friction between the ball and the ground is $\mu_s = \mu_k = \mu$. Determine the time the ball begins to roll without slipping.

Problem 4I. A hoop of radius r is launched to the right at initial speed v. As it is launched, it is also spun counterclockwise with spin angular velocity $\frac{3v}{r}$. The coefficient of kinetic friction between the ground and the hoop is μ_k.

(a) How long does it take the hoop to return to its starting position?

(b) If the hoop were replaced by a uniform disk, how would the answer change?

Problem 4J. A spool is made of a cylinder with a thin disc attached to either end of the cylinder as shown in Figure 4.9. The cylinder has radius r and the discs each have radius $R > r$. A string is attached to the cylinder and wound around the cylinder a few times. At what angle above the horizontal can the string be pulled so that the spool will slip without rotating?

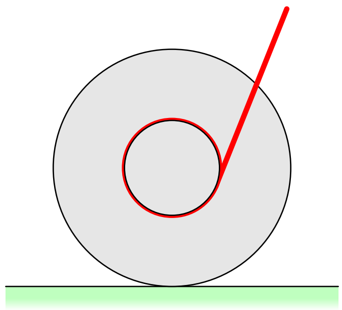

Figure 4.9: A cylindrical spool, side view.

Problem 4K. A uniform solid circular disk of mass m is on a flat, frictionless horizontal table. The center of mass of the disk is at rest and the disk is spinning with spin angular velocity ω. A stone, modeled as a point object also of mass m, is placed on the edge of the disk, with zero initial velocity relative to the table. A rim built into the disk constrains the stone to slide, with friction, along the disk's edge. After the stone stops sliding with respect to the disk, what is the spin angular speed of rotation of the disk and the stone together?

5 Fictitious forces

A **fictitious force** is a force that arises when working in non-inertial reference frames. In a non-inertial reference frame, Newton's second law does not hold, but the addition of fictitious forces into the model can resolve the discrepancy. In other words, a fictitious force is a force that appears to be acting on an object when its motion is described in a non-inertial frame of reference. Such a force does not actually exist in an inertial frame.

Remark 5.0.1 — Newton's third law is no longer satisfied when fictitious forces are added into the model; in particular, fictitious forces do not have a corresponding force in non-inertial reference frames. However, Newton's second law remains valid; after all, fictitious forces serve as a correction to Newton's second law, allowing it to hold in non-inertial reference frames.

§5.1 Rectilinear acceleration

Prototypical example for this section: an accelerating car.

Suppose a non-inertial reference frame is linearly accelerating with acceleration $\mathbf{a}$ with respect to an inertial reference frame. Then there is a fictitious force acting on all objects in the opposite direction of the acceleration, and the fictitious force on an object with mass m is $-m\mathbf{a}$ by Newton's second law.

Example 5.1.1 (Accelerating car)
Suppose a car is speeding up. Passengers in the car will feel themselves being pushed back in their seats; that is, in the reference frame fixing the car, it appears that there is a mysterious backwards-pointing force acting on all objects in the car. This can be explained in an inertial reference frame because objects tend to move with constant velocity, but in the non-inertial reference frame this tendency is said to be caused by a fictitious force.

§5.2 Centrifugal force

Prototypical example for this section: a rotating amusement park ride.

The most common fictitious force is the **centrifugal force**, which arises when considering a rotating reference frame, such as the one fixing the Earth.

Example 5.2.1 (Gravitron)
Consider an amusement park ride in which passengers lean against the wall of a circular contraption which rotates at a high speed. In the non-inertial reference frame fixing the contraption, passengers appear to be pushed back into their seats. In the inertial reference frame, this mysterious force can be explained by an object's tendency to move in a straight line.

Strictly speaking, the centrifugal force is the apparent force on the object, assuming the object's velocity is at rest in the rotating reference frame and the reference frame rotates at a constant angular velocity.

Theorem 5.2.2 (Centrifugal force formula)
The centrifugal force that appears to act on an object with mass m in a rotating reference frame with angular velocity $\boldsymbol{\omega}$ has magnitude

$$\|\mathbf{F}\| = m \|\boldsymbol{\omega}\|^2 r$$

and points directly away from the axis of rotation, where r is the distance from the object to the axis of rotation.

Proof. In the inertial reference frame, the object is moving uniformly around a circle with radius r at an angular speed of $\|\boldsymbol{\omega}\|$. By the centripetal force formula, a force of $\|\boldsymbol{\omega}\|^2 r$ point towards the axis of rotation is needed to keep the object in this circle. However, in the rotating reference frame, the object appears to be stationary, so there needs to be a fictitious force of magnitude $\|\boldsymbol{\omega}\|^2 r$ pointing away from the axis of rotation, which is the centrifugal force. □

§5.3 Coriolis force

The **Coriolis force** is a fictitious force that appears in a rotating reference frame when an object has a nonzero velocity in the reference frame.

Example 5.3.1 (Bullet)
Consider a bullet shot from the Northern hemisphere over a very long distance towards the North Pole of the Earth. The bullet retains some horizontal velocity from the location it was shot at, but as it moves northwards, the ground under it moves at a slower rate, so from an observer on the Earth, the bullet appears to be deflected. This apparent deflection is caused by the Coriolis force.

Strictly speaking, the Coriolis force is the apparent non-centrifugal force on an object viewed in a rotating reference frame with constant angular velocity.

Before deriving the Coriolis force formula, however, the following lemma is necessary.

Lemma 5.3.2
Let $\mathbf{x}$ be the trajectory of a particle in an inertial frame of reference, and let $\mathbf{x}'$ be its trajectory in a rotating frame of reference fixing the origin with angular velocity $\boldsymbol{\omega}$. Then

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{v}'$$

where $\mathbf{v}'$ is the velocity of the particle in the rotating reference frame.

Proof. By the product rule,

$$\begin{aligned}
\mathbf{v} = \frac{d\mathbf{x}}{dt} &= \frac{d}{dt}\left[\mathrm{proj}_{\hat{\mathbf{i}}'}(\mathbf{x})\hat{\mathbf{i}}' + \mathrm{proj}_{\hat{\mathbf{j}}'}(\mathbf{x})\hat{\mathbf{j}}' + \mathrm{proj}_{\hat{\mathbf{k}}'}(\mathbf{x})\hat{\mathbf{k}}'\right] \\
&= \frac{d}{dt}\left[\mathrm{proj}_{\hat{\mathbf{i}}'}(\mathbf{x})\right]\hat{\mathbf{i}}' + \frac{d}{dt}\left[\mathrm{proj}_{\hat{\mathbf{j}}'}(\mathbf{x})\right]\hat{\mathbf{j}}' + \frac{d}{dt}\left[\mathrm{proj}_{\hat{\mathbf{k}}'}(\mathbf{x})\right]\hat{\mathbf{k}}' \\
&\qquad + \mathrm{proj}_{\hat{\mathbf{i}}'}(\mathbf{x})\frac{d\hat{\mathbf{i}}'}{dt} + \mathrm{proj}_{\hat{\mathbf{j}}'}(\mathbf{x})\frac{d\hat{\mathbf{j}}'}{dt} + \mathrm{proj}_{\hat{\mathbf{k}}'}(\mathbf{x})\frac{d\hat{\mathbf{k}}'}{dt} \\
&= \frac{d\mathbf{x}'}{dt} + \mathrm{proj}_{\hat{\mathbf{i}}'}(\mathbf{x})(\boldsymbol{\omega}\times\hat{\mathbf{i}}) + \mathrm{proj}_{\hat{\mathbf{j}}'}(\mathbf{x})(\boldsymbol{\omega}\times\hat{\mathbf{j}}) + \mathrm{proj}_{\hat{\mathbf{k}}'}(\mathbf{x})(\boldsymbol{\omega}\times\hat{\mathbf{k}}) \\
&= \frac{d\mathbf{x}'}{dt} + \boldsymbol{\omega}\times(\mathrm{proj}_{\hat{\mathbf{i}}'}(\mathbf{x})\hat{\mathbf{i}}' + (\mathrm{proj}_{\hat{\mathbf{j}}'}(\mathbf{x})\hat{\mathbf{j}}' + (\mathrm{proj}_{\hat{\mathbf{k}}'}(\mathbf{x})\hat{\mathbf{k}}') \\
&= \frac{d\mathbf{x}'}{dt} + \boldsymbol{\omega}\times\mathbf{x}',
\end{aligned}$$

using Corollary 1.9.2. ☐

Theorem 5.3.3 (Coriolis force formula)
The Coriolis force that appears to act on an object with mass m in a rotating reference frame with angular velocity $\boldsymbol{\omega}$ is

$$\mathbf{F} = -2m\boldsymbol{\omega}\times\mathbf{v}' = 2m\mathbf{v}'\times\boldsymbol{\omega},$$

where $\mathbf{v}'$ is the velocity of the object in the rotating reference frame.

Proof. Suppose the object has trajectory $\mathbf{x}$ in the inertial reference frame, and suppose the rotating reference frame fixes the origin and has angular velocity $\boldsymbol{\omega}$. Additionally, let $\mathbf{x}'$ be the trajectory of the object with respect to the rotating reference frame.

By Lemma 5.3.2,

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} = \mathbf{v}' + \boldsymbol{\omega}\times\mathbf{x}';$$

taking the derivative and applying the lemma once again yields

$$\mathbf{a} = \left(\frac{d\mathbf{v}'}{dt} + \boldsymbol{\omega}\times\mathbf{v}'\right) + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{x}' + \boldsymbol{\omega}\times\frac{d\mathbf{x}'}{dt} = \mathbf{a}' + \boldsymbol{\omega}\times\mathbf{v}' + \boldsymbol{\alpha}\times\mathbf{x}' + \boldsymbol{\omega}\times(\mathbf{v}' + \boldsymbol{\omega}\times\mathbf{x}'),$$

so rearranging gives

$$\mathbf{a}' = \mathbf{a} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x}') - 2\boldsymbol{\omega}\times\mathbf{v}' - \boldsymbol{\alpha}\times\mathbf{x}'.$$

Multiplying both sides by m gives

$$\mathbf{F}' = \mathbf{F} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x}') - 2m\boldsymbol{\omega}\times\mathbf{v}' - m\boldsymbol{\alpha}\times\mathbf{x}'.$$

The net fictitious force acting on the particle is thus

$$-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x}') - 2m\boldsymbol{\omega}\times\mathbf{v}' - m\boldsymbol{\alpha}\times\mathbf{x}'.$$

The centrifugal force, from this expression, is equal to

$$m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x}'),$$

since this vector points directly away from the axis of rotation and has magnitude $\|\boldsymbol{\omega}\|^2 r$. Additionally, $\boldsymbol{\alpha} = \mathbf{0}$. The remaining term is the Coriolis force, since it depends on the velocity of the object. ☐

§5.4 Euler force

The **Euler force** is the component of the apparent force that only appears when there is a nonzero angular acceleration of the rotating reference frame.

Example 5.4.1 (Merry-go-round)
Consider a child on a merry-go-round. When the merry-go-round starts accelerating, the child will appear to experience a force pushing them backwards in the frame fixing the merry-go-round; this force is the Euler force.

Strictly speaking, the Euler force is the non-centrifugal, non-Coriolis component of the total fictitious force on an object.

Theorem 5.4.2 (Euler force formula)
The Euler force that appears to act on an object with mass m in a rotating reference frame with angular velocity $\boldsymbol{\omega}$ is

$$\mathbf{F} = -m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} = -m\boldsymbol{\alpha} \times \mathbf{r}.$$

Proof. From the derivation of the centrifugal force, the net fictitious force acting on the object is

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}') - 2m\boldsymbol{\omega} \times \mathbf{v}' - m\boldsymbol{\alpha} \times \mathbf{x}'.$$

The first two terms are the centrifugal and the Coriolis force, respectively, so the third force must be the Euler force since it depends on angular acceleration. □

§5.5 A few harder problems to think about

Problem 5A. A cylindrical space station produces "artificial gravity" by rotating at a constant angular velocity along its axis, as shown in Figure 5.1. Consider working in the reference frame rotating with the space station. In this frame, an astronaut is initially at rest standing on the floor, facing in the direction that the space station is rotating. The astronaut jumps up vertically relative to the floor of the space station, with an initial speed less than that of the speed of the floor. Just after leaving the floor, the motion of the astronaut, relative to the space station floor, determine

(a) the direction of the component of acceleration perpendicular to the floor, and

(b) where the astronaut lands relative to the point they jumped from.

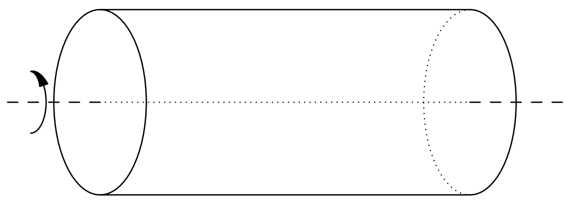

Figure 5.1: A cylindrical space station.

Problem 5B. A man standing at 30° latitude fires a bullet northward as a speed of $200\,\frac{\mathrm{m}}{\mathrm{s}}$. The radius of the Earth is 6371 km. What is the sideways deflection of the bullet after traveling 100 m?

Problem 5C. A child in a circular, rotating space station tosses a ball in such a way so that once the station has rotated through one half rotation, the child catches the ball. From the child's point of view, plot the trajectory of the ball.

Problem 5D. Alice and Bethany stand side by side on the Earth's equator. If Alice jumps directly upward, in her frame of reference, to a small height h much less than the radius of the Earth, she will land a distance D to the west of Bethany. If Alice had instead jumped to a height $2h$, how far to the west of Bethany would she land? Neglect air resistance.

6 Energy

§6.1 Work

Prototypical example for this section: pushing a box on the ground.

The **work** done by a force on a particle over a path C is the line integral of the dot product of the force and its infinitesimal displacement; that is,

$$W = \int_C \mathbf{F} \cdot d\mathbf{x} = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}\, dt.$$

Work is a scalar quantity measured in **joules**, abbreviated "J".

Remark 6.1.1 — A joule is defined as the work required to exert a force of one newton through a displacement of one meter.

One joule can be expressed as

$$1\,\mathrm{J} = 1\,\mathrm{N} \cdot \mathrm{m} = 1\,\frac{\mathrm{kg} \cdot \mathrm{m}^2}{\mathrm{s}^2}.$$

If a force always has a component in the direction of the path, it does positive work. If a force always has a component opposite the direction of the path, it does negative work. If a force is always perpendicular to the path, it does no work.

When the force $\mathbf{F}$ is constant and the angle between the force and the infinitesimal displacement $d\mathbf{s}$ is always θ, then the work done is given by

$$W = \int_C \mathbf{F} \cdot d\mathbf{s} = \mathbf{F} \cdot \int_C d\mathbf{s} = \mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\|\,\|\mathbf{s}\| \cos\theta.$$

Exercise 6.1.2. A waiter is carrying a tray such that both the tray's height and the tray's velocity is constant. Explain why the waiter does no work on the tray.

The **work** done by a force on a rigid body is the sum of the work done by the force on each point particle. Work can be decomposed into a linear component and a rotational component.

Corollary 6.1.3 (Work on rigid body)
The work done by a force on a rigid body is

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}_{\mathrm{cm}} + \boldsymbol{\tau} \cdot \boldsymbol{\omega}\, dt,$$

where $\mathbf{F}$ is the sum of the individual forces on each point particle caused by the force and $\boldsymbol{\tau}$ is the sum of the individual torques on each point particle caused by the force.

Proof. Let $\mathbf{F}_i$ be the force on particle i caused by the force in consideration. By definition,

$$\begin{aligned}
W &= \sum_i \int_{t_1}^{t_2} \mathbf{F}_i \cdot \mathbf{v}_i \, dt \\
&= \sum_i \int_{t_1}^{t_2} \mathbf{F}_i \cdot (\mathbf{v}_{\text{cm}} + \boldsymbol{\omega} \times (\mathbf{x}_i - \mathbf{x}_{\text{cm}})) \, dt \\
&= \int_{t_1}^{t_2} \left[\sum_i [\mathbf{F}_i \cdot \mathbf{v}_{\text{cm}}] + \sum_i [\mathbf{F}_i \cdot \boldsymbol{\omega} \times (\mathbf{x}_i - \mathbf{x}_{\text{cm}})] \right] dt \\
&= \int_{t_1}^{t_2} \left[\mathbf{v}_{\text{cm}} \cdot \sum_i [\mathbf{F}_i] + \boldsymbol{\omega} \cdot \sum_i [\mathbf{F}_i \cdot (\mathbf{x}_i - \mathbf{x}_{\text{cm}})] \right] dt \\
&= \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}_{\text{cm}} + \boldsymbol{\tau} \cdot \boldsymbol{\omega} \, dt.
\end{aligned}$$

□

§6.2 Conservative forces

Prototypical example for this section: gravity near Earth.

A **conservative force** is a force with the property that the total work needed to move a particle from one point to another point is independent of the path taken, and a **non-conservative force** is a force that does not satisfy this property.

If a force is conservative, given a particle with fixed mass, a numerical value U measured in joules can be assigned to each point, known as the **potential** at that point for the particle. This potential has the property that the total work needed to move the particle from one point to another point is the negative of the difference between the potentials of the two points; that is,

$$W = -\Delta U = -(U_2 - U_1).$$

The negative sign is a result of the convention that work done opposing a force increases potential energy, while work done in the direction of the force decreases potential energy.

Generally, the assignment of potentials to every point is not unique, since a constant can be added to every point without changing the difference in potentials between any two points. Therefore, some convenient location is assigned, by convention, a potential of zero. This uniquely determines the potential of all other points.

Gravity is an example of a conservative force, while friction is not conservative. Since friction always acts against the direction of motion and has a constant magnitude, the work done by a frictional force will be proportional to path length – in particular, it is not independent of path.

§6.3 Gravitational potential energy near Earth

Prototypical example for this section: a rock on top of a hill.

Gravity near Earth's surface is conservative because the total work needed to move a particle m from one point $\mathbf{r}_1$ to another point $\mathbf{r}_2$ along a path C is

$$\int_C \left(-mg\hat{\mathbf{k}}\right) \cdot d\mathbf{s} = -mg\hat{\mathbf{k}} \cdot \int_C d\mathbf{s} = -mg\hat{\mathbf{k}} \cdot (\mathbf{r}_2 - \mathbf{r}_1) = -(mgh_2 - mgh_1),$$

where h_1 and h_2 are the signed magnitudes of the projections of $\mathbf{r}_1$ and $\mathbf{r}_2$ onto $\hat{\mathbf{k}}$; that is, h_1 and h_2 are the signed distances from $\mathbf{r}_1$ and $\mathbf{r}_2$ to the ground. Thus, a potential at each point can be assigned, known as the **gravitational potential energy**, measured in joules.

By convention, the gravitational potential energy of a point particle on the ground – which is assumed to be flat – is zero, so the gravitational potential energy of a particle with mass m is

$$U_g = mgh,$$

where h is the height of the particle above the ground.

§6.4 Spring potential energy

Prototypical example for this section: the energy stored in a wind-up toy.

Because the spring force only depends on the displacement from its equilibrium state and springs operate one-dimensionally, the spring force is conservative. Thus, a potential at each point can be assigned, known as the **spring potential energy**, measured in joules. The total work needed to move a spring from a position $\mathbf{x}_1$ to $\mathbf{x}_2$ is

$$\int_{\mathbf{x}_1}^{\mathbf{x}_2} (-k\mathbf{x}) \cdot d\mathbf{x} = -k \int_{x_1}^{x_2} x \cdot dx = -k \left(\frac{1}{2} x_2^2 - \frac{1}{2} x_1^2 \right).$$

By convention, the spring potential energy of a spring in equilibrium position is zero, so the spring potential energy of a spring stretched or compressed with a displacement of magnitude x is

$$U_s = \frac{1}{2} kx^2.$$

§6.5 Kinetic energy

Prototypical example for this section: a moving car.

The **kinetic energy** of a point particle is the work needed to accelerate it from rest to its desired velocity.

Theorem 6.5.1 (Kinetic energy formula)
The kinetic energy of a point particle with mass m and velocity $\mathbf{v}$ is

$$E_k = \frac{1}{2} m\mathbf{v} \cdot \mathbf{v} = \frac{1}{2} mv^2.$$

Proof. suppose the particle accelerates from rest to velocity $\mathbf{v}$ between times t_1 and t_2. Since $\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$, it follows from the definition of kinetic energy that

$$E_k = \int_{t_1}^{t_2} \left(m \frac{d\mathbf{v}}{dt} \right) \cdot \mathbf{v} dt = \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt} [\mathbf{v} \cdot \mathbf{v}] \, dt = \frac{1}{2} m\mathbf{v} \cdot \mathbf{v} - \frac{1}{2} m\mathbf{0} \cdot \mathbf{0} = \frac{1}{2} mv^2.$$

□

The **kinetic energy** of a rigid body is the sum of the kinetic energies of the point particles that comprise the rigid body.

Corollary 6.5.2 (Kinetic energy decomposition)

Let ω be the spin angular speed of a rigid body and let $I_{\boldsymbol{\omega}}$ be the moment of inertia around the line through its center of mass parallel to its spin angular velocity $\boldsymbol{\omega}$. Then

$$E_k = \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}I_{\boldsymbol{\omega}}\omega^2.$$

Proof. Let θ_i be the angle between $\mathbf{x}_i - \mathbf{x}_{\text{cm}}$ and ω, and let r_i be the distance from $\mathbf{x}_i$ to the line through the center of mass parallel to $\boldsymbol{\omega}$. By expansion,

$$\begin{aligned}
E_k &= \sum_i \frac{1}{2}m_i\mathbf{v}_i \cdot \mathbf{v}_i \\
&= \sum_i \left[\frac{1}{2}m_i\mathbf{v}_{\text{cm}}\cdot\mathbf{v}_{\text{cm}}\right] + \sum_i \left[m_i\mathbf{v}_{\text{cm}}\cdot(\mathbf{v}_i - \mathbf{v}_{\text{cm}})\right] + \sum_i \left[\frac{1}{2}m_i\left(\mathbf{v}_i - \mathbf{v}_{\text{cm}}\right)\cdot(\mathbf{v}_i - \mathbf{v}_{\text{cm}})\right] \\
&= \frac{1}{2}m\mathbf{v}_{\text{cm}} \cdot \mathbf{v}_{\text{cm}} + \mathbf{v}_{\text{cm}} \cdot \sum_i \left[m_i(\mathbf{v}_i - \mathbf{v}_{\text{cm}})\right] + \sum_i \left[\frac{1}{2}m_i \left\|\boldsymbol{\omega} \times (\mathbf{x}_i - \mathbf{x}_{\text{cm}})\right\|^2\right] \\
&= \frac{1}{2}m\left\|\mathbf{v}_{\text{cm}}\right\|^2 + \mathbf{v}_{\text{cm}} \cdot \mathbf{0} + \sum_i \left[\frac{1}{2}m_i\left(\sin\theta_i \left\|\mathbf{x}_i - \mathbf{x}_{\text{cm}}\right\|\right)^2 \left\|\boldsymbol{\omega}\right\|^2\right] \\
&= \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}\left\|\boldsymbol{\omega}\right\|^2 \sum_i \left[m_i r_i^2\right] \\
&= \frac{1}{2}mv_{\text{cm}}^2 + I_{\boldsymbol{\omega}}\omega^2.
\end{aligned}$$

□

The components above are known as the **translational kinetic energy** and the **rotational kinetic energy** of a rigid body; that is,

$$E_t = \frac{1}{2}mv_{\text{cm}}^2$$

and

$$E_r = \frac{1}{2}I_{\boldsymbol{\omega}}\omega^2 = \frac{1}{2}\left(\frac{\boldsymbol{\omega}}{\omega}\cdot\mathbf{I}_{\text{cm}}\frac{\boldsymbol{\omega}}{\omega}\right)\omega^2 = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}_{\text{cm}}\boldsymbol{\omega} = \frac{1}{2} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}_s.$$

Intuitively, the translational kinetic energy is equal to the kinetic energy of the rigid body assuming that all its mass was concentrated at its center of mass, and the rotational kinetic energy is equal to the kinetic energy of the rigid body in the reference frame centered at its center of mass.

§6.6 Work-energy principle

The **work-energy principle** relates the work done on a particle with the change in its kinetic energy.

Theorem 6.6.1 (Work-energy principle for point particles)

The work done by all forces acting on a particle – that is, the work done by the net force acting on the particle – equals the change in its kinetic energy; that is,

$$W = \Delta E_k = (E_k)_2 - (E_k)_1 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2.$$

Proof. Suppose a force $\mathbf{F}$ causes an object to change from a speed $\mathbf{v}_1$ to a speed $\mathbf{v}_2$ between times t_1 and t_2. Similarly to the derivation of the kinetic energy formula, the total work done on the object is

$$\begin{aligned} W &= \int_{t_1}^{t_2} \left(m \frac{d\mathbf{v}}{dt} \right) \cdot \mathbf{v} dt = \frac{1}{2} m \int_0^t \frac{d}{dt} \left[\mathbf{v} \cdot \mathbf{v} \right] dt \\ &= \frac{1}{2} m \mathbf{v}_2 \cdot \mathbf{v}_2 - \frac{1}{2} m \mathbf{v}_1 \cdot \mathbf{v}_1 = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2. \end{aligned}$$

□

The work-energy principle also holds for rigid bodies.

Theorem 6.6.2 (Work-energy principle)
The work done by all forces acting on a rigid body equals the change in its kinetic energy; that is,

$$W = \Delta E_k = (E_k)_2 - (E_k)_1 = \left(\frac{1}{2} m v_2^2 - \frac{1}{2} I_{\omega_2} \omega_2^2 \right) - \left(\frac{1}{2} m v_1^2 - \frac{1}{2} I_{\omega_1} \omega_1^2 \right).$$

Proof. Since $\mathbf{I}_{\text{cm}}$ is symmetric, $\boldsymbol{\omega} \times \mathbf{I}_{\text{cm}} \boldsymbol{\alpha} = \boldsymbol{\alpha} \cdot \mathbf{I}_{\text{cm}} \boldsymbol{\omega}$. Thus,

$$\boldsymbol{\omega} \cdot \boldsymbol{\tau} = \boldsymbol{\omega} \cdot (\mathbf{I}_{\text{cm}} \boldsymbol{\alpha} + \boldsymbol{\omega} \times \mathbf{L}_s) = \boldsymbol{\omega} \cdot \mathbf{I}_{\text{cm}} \boldsymbol{\alpha} + \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \mathbf{L}_s) = \boldsymbol{\alpha} \cdot \mathbf{I}_{\text{cm}} \boldsymbol{\omega} + \mathbf{L}_s \cdot (\boldsymbol{\omega} \times \boldsymbol{\omega}) = \boldsymbol{\alpha} \cdot \mathbf{L}_s,$$

by Euler's rotation equation, so

$$\frac{d}{dt} \left[\mathbf{L}_s \cdot \boldsymbol{\omega} \right] = \boldsymbol{\tau} \cdot \boldsymbol{\omega} + \mathbf{L}_s \cdot \boldsymbol{\alpha} = 2 \boldsymbol{\tau} \cdot \boldsymbol{\omega}$$

by the product rule. Hence,

$$\begin{aligned} W &= \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}_{\text{cm}} + \boldsymbol{\tau} \cdot \boldsymbol{\omega} \, dt \\ &= \int_{t_1}^{t_2} m \mathbf{a}_{\text{cm}} \cdot \mathbf{v}_{\text{cm}} \, dt + \int_{t_1}^{t_2} \boldsymbol{\tau} \cdot \boldsymbol{\omega} \, dt \\ &= \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt} \left[\mathbf{v}_{\text{cm}} \cdot \mathbf{v}_{\text{cm}} \right] \, dt + \frac{1}{2} \int_{t_1}^{t_2} \frac{d}{dt} \left[\mathbf{L}_s \cdot \boldsymbol{\omega} \right] dt \\ &= \left(\frac{1}{2} m \mathbf{v}_2 \cdot \mathbf{v}_2 - \frac{1}{2} m \mathbf{v}_1 \cdot \mathbf{v}_1 \right) + \left(\frac{1}{2} \mathbf{I}_2 \boldsymbol{\omega}_2 \cdot \boldsymbol{\omega}_2 - \frac{1}{2} \mathbf{I}_1 \boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_1 \right) \\ &= \left(\frac{1}{2} m v_2^2 - \frac{1}{2} I_{\omega_2} \omega_2^2 \right) - \left(\frac{1}{2} m v_1^2 - \frac{1}{2} I_{\omega_1} \omega_1^2 \right). \end{aligned}$$

□

Remark 6.6.3 — In particular, the kinetic energy of a rigid body without any net forces or net torques acting on it is constant throughout its trajectory.

Exercise 6.6.4. An object is thrown up into the air. If air resistance is not ignored, explain why its speed upon return is always smaller than its initial speed.

§6.7 Mechanical energy

Prototypical example for this section: a roller coaster in the middle of its trajectory.

The mechanical energy of an object is the sum of its potential energies and its kinetic energy.

Theorem 6.7.1 (Conservation of mechanical energy)
The mechanical energy of a system is constant as long as the non-conservative forces acting on it always act perpendicular to its direction of motion and exert no net torque.

Proof. By the work-energy principle, a change in kinetic energy corresponds to work done on the system by a force. Because the non-conservative forces are assumed to exert no net torque and always act perpendicularly to the direction of motion, the non-conservative forces do no work on the system by Corollary 6.1.3.

Now, the work-energy principle states that the change in kinetic energy equals the work done on the object. Since the non-conservative forces do no work, only the work done by conservative forces needs to be considered. For each conservative force, the work done by the conservative force is equal to the decrease in potential energy associated with the conservative force, by definition. Hence the work done – which is the increase in potential energy – equals the decrease in potential energy. $\square$

The normal force and the tension force are non-conservative, but when they act perpendicularly to the direction of motion – such as in the case of an object moving on a frictionless surface, or a string holding a swinging object – they never do work on an object.

In a similar vein, we also have the following result.

Corollary 6.7.2 (Rolling without slipping conserves mechanical energy)
If an object is rolling without slipping and the only forces acting on the object are friction, gravity, and the normal force, then mechanical energy is conserved.

Proof. When rolling without slipping, the friction force does no work because it is exerted on a point on the rigid body with no velocity. $\square$

§6.8 Power

Power is the rate at which work is done; that is,

$$P = \frac{dW}{dt}.$$

Power is measured in **joules per second**, also known as **watts**, abbreviated "W". One watt can be expressed as

$$1\text{ W} = 1\,\frac{\text{J}}{\text{s}} = 1\,\frac{\text{kg}\cdot\text{m}^2}{\text{s}^3}.$$

§6.9 Elastic collisions

Prototypical example for this section: a bouncy ball.

An **elastic collision** is one in which the total mechanical energy of the system is constant throughout the collision.

Corollary 6.9.1 (Elastic bouncing)

If an object bounces elastically off of a surface, then its speed before the collision equals its speed after the collision, and its angle of incidence equals its angle of reflection.

Proof. The easiest way to see this is to note that the component of velocity parallel to the surface is fixed because the normal force the wall exerts on the object is perpendicular to this component of velocity and thus cannot change it. Since energy is conserved, its speed is constant, so the component of velocity perpendicular to the plane of the fixed object must have the same magnitude before and after the collision, and its sign must flip. □

One-dimensional elastic collision problems are commonly found in an introductory physics course.

Theorem 6.9.2 (One-dimensional elastic collision)

Suppose a block of mass m_1 moving at signed speed v_1 collides elastically with a block of mass m_2 moving at signed speed v_2. Then their speeds after the collision are

$$\begin{aligned} v_1' &= \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2 \\ v_2' &= \frac{2m_1}{m_1 + m_2} v_1 + \frac{m_2 - m_1}{m_1 + m_2} v_2. \end{aligned}$$

Proof. By conservation of momentum,

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'.$$

By conservation of energy,

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2.$$

Solving for v_1' and v_2' gives the solutions

$$v_1' = v_1, \quad v_2' = v_2$$

and

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2, \quad v_2' = \frac{2m_1}{m_1 + m_2} v_1 + \frac{m_2 - m_1}{m_1 + m_2} v_2.$$

Since the blocks are assumed to collide, the first solution is not valid, so the second solution must be the resulting speeds of the blocks. □

Exercise 6.9.3. Show that the elasticity of a collision does not depend on the reference frame as long as the reference frame is inertial.

§6.10 Inelastic collisions

Prototypical example for this section: a car crash.

An **inelastic collision** is a collision in which kinetic energy is not conserved. Most everyday collisions are inelastic.

A **perfectly inelastic collision** is one in which the maximum amount of kinetic energy is lost. Perfectly inelastic collisions can be characterized as those for which the objects stick together after the collision.

Corollary 6.10.1 (Perfectly inelastic collision)
In a perfectly inelastic collision, the resulting velocities of every particle after the collision is equal.

Proof. By conservation of momentum,

$$\sum_i m_i \mathbf{v}_i$$

is constant. Suppose that two particles m_1 and m_2 after the collision have different velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. If their velocities were both replaced by the velocity of their center of mass $\frac{m_1\mathbf{v}_1+m_2\mathbf{v}_2}{m_1+m_2}$ (which is possible because momentum is conserved), then the kinetic energy cannot increase, since

$$\frac{1}{2}m_1\left\|\frac{m_1\mathbf{v}_1+m_2\mathbf{v}_2}{m_1+m_2}\right\|^2+\frac{1}{2}m_2\left\|\frac{m_1\mathbf{v}_1+m_2\mathbf{v}_2}{m_1+m_2}\right\|^2\leq\frac{1}{2}m_1\left\|\mathbf{v}_1\right\|^2+\frac{1}{2}m_2\left\|\mathbf{v}_2\right\|^2,$$

as the inequality is equivalent to

$$\frac{1}{2}\frac{m_1m_2}{m_1+m_2}\left\|\mathbf{v}_2-\mathbf{v}_1\right\|^2\geq 0.$$

Since the inequality is tight only when $\mathbf{v}_1=\mathbf{v}_2$, the least possible kinetic energy after the collision occurs when the velocity of every point particle is the same. □

§6.11 A few harder problems to think about

Problem 6A. An escalator can carry passengers up a vertical distance of 10 m in 30 s. A mischievous person of mass 50 kg walks down the up-escalator so that they stay in place with respect to the building. If the child does this for 30 s, determine the total work the child performs on the escalator in the frame of the building.

Problem 6B. A cylindrical bucket of negligible mass has radius r and height h, and is open at the top. It is submerged in water of density ρ, with its top a distance H below the surface, as shown in Figure 6.1. How much work is needed to pull the bucket slowly up so that its bottom is just above the lake surface?

Problem 6C. An object is thrown at a speed v and angle θ off a building with height h. Determine its impact speed, in terms of v.

Problem 6D. A ball is launched straight toward the ground from height h. When it bounces off the ground, it loses half of its kinetic energy. It reaches a maximum height of $2h$ before falling back to the ground again. What was the initial speed of the ball?

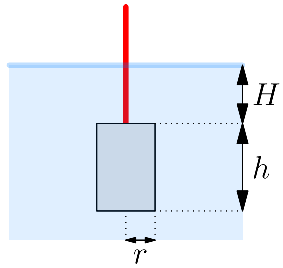

Figure 6.1: A submerged bucket.

Problem 6E. A uniform block of mass 10 kg is released at rest from the top of an incline with length 10 m and inclination 30°, and slides to the bottom. The coefficients of static and kinetic friction are $\mu_\mathrm{s} = \mu_\mathrm{k} = 0.1$. How much energy is dissipated due to friction?

Problem 6F. A 3.0 kg moving at 40 $\frac{\mathrm{m}}{\mathrm{s}}$ to the right collides with and sticks to a 2.0 kg mass traveling at 20 $\frac{\mathrm{m}}{\mathrm{s}}$ to the right. Determine the kinetic energy of the system after the collision.

Problem 6G. A mass of $3m$ moving at a speed v collides with a mass of m moving directly towards it, also with a speed v. If the collision is completely elastic, the total kinetic energy after the collision is K_e. If the two masses stick together, the total kinetic energy after the collision is K_s. Determine $\frac{K_e}{K_s}$.

Problem 6H. Three boxes A, B, and C lie along a straight line on a horizontal frictionless surface, as shown in Figure 6.2. Box A is initially moving to the right with speed v while the other two boxes are initially at rest. If all collisions are elastic and the masses of the boxes can be chosen freely, what is smallest speed that must be greater than all possible final speeds of box C?

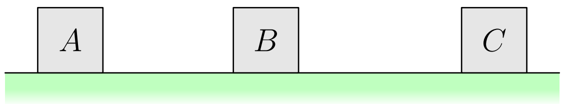

Figure 6.2: Three boxes on a horizontal frictionless surface.

Problem 6I. A point particle is attached to the end of the massless rod of length l. The other end of the rod is attached to a frictionless pivot. The object is raised so that its height is $0.8l$ above the pivot, as shown in Figure 6.3. After the object is released from rest, what is the tension in the rod when it is horizontal?

Problem 6J. A block of mass m is launched horizontally onto a curved wedge of mass M at a velocity v, as shown in 6.4. What is the maximum height reached by the block after it shoots off the vertical segment of the wedge? Assume all surfaces are frictionless.

Problem 6K. Two springs of spring constants k_1 and k_2, respectively, are connected in series and stretched, as shown in Figure 6.5. What is the ratio of their potential energies, $\frac{U_1}{U_2}$?

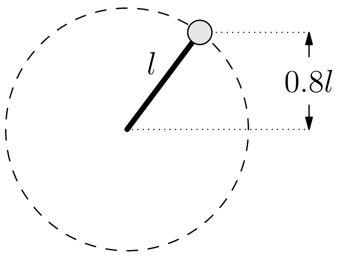

Figure 6.3: A mass pivoting around a rod.

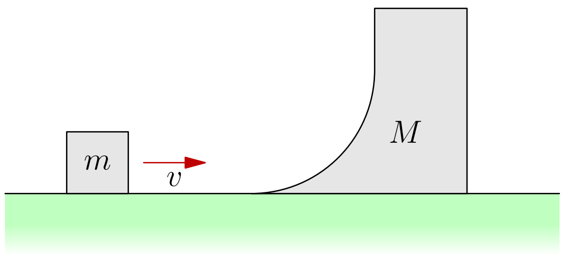

Figure 6.4: A block launched onto a curved wedge.

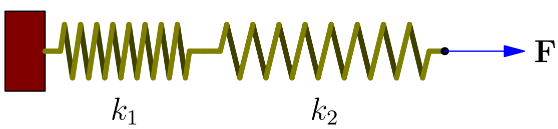

Figure 6.5: Two springs connected in series.

Problem 6L. A pogo stick is modeled as a massless spring of spring constant k attached to the bottom of a block of mass m, as shown in Figure 6.6. The pogo stick is dropped wit the spring pointing downward and hits the ground with speed v. At the moment of the collision, the free end of the spring sticks permanently to the ground. Determine the maximum speed of the block.

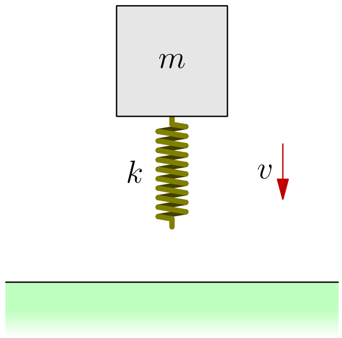

Figure 6.6: A pogo stick.

7 Periodic motion

A system is in **periodic motion** if its motion repeats after some fixed amount of time, known as the system's **period**. Periodicity is measured in seconds. The **frequency** of a system in periodic motion is the reciprocal of its period and represents how often the system's motion oscillates per second. Frequency is measured in inverse seconds $\frac{1}{\text{s}}$, also known as **hertz**, abbreviated Hz.

§7.1 Simple harmonic oscillation

Prototypical example for this section: A mass attached to a spring.

Consider an object positioned in such a way that the force acting on it is always proportional to and opposite in direction from its displacement, such as a block on a surface attached to a spring, or a block hanging from a spring. Such a system is called a **harmonic oscillator**.

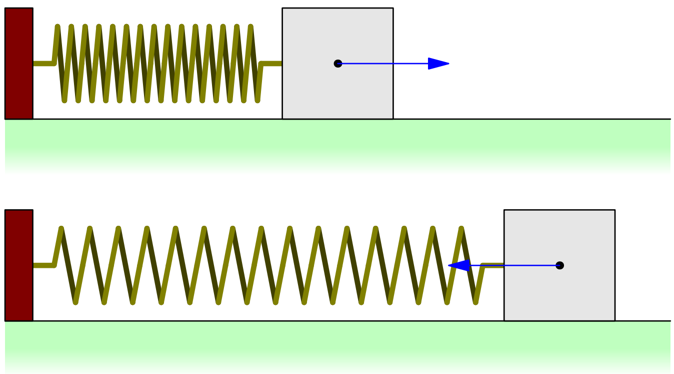

Figure 7.1: A harmonic oscillator.

A system is in **simple harmonic oscillation** if no other forces, such as friction, act on the object.

Theorem 7.1.1 (Period of simple harmonic oscillation)
Suppose an object with mass m is in simple harmonic oscillation and the constant of proportionality between the restoring force and the object's displacement is k. Then the object oscillates with period

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Proof. By Hooke's law and Newton's second law,

$$-k\mathbf{x} = \mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{x}}{dt^2}$$

which gives the differential equation

$$\frac{d^2\mathbf{x}}{dt^2} + \frac{k}{m}\mathbf{x} = 0.$$

The characteristic equation of this second-order linear differential equation, $r^2 + \frac{k}{m} = 0$, has roots $\pm i\sqrt{\frac{k}{m}}$, so the general solution is

$$\mathbf{x} = c_1 \cos\left(\sqrt{\frac{k}{m}}t + c_2\right)\hat{\mathbf{i}} + c_3 \cos\left(\sqrt{\frac{k}{m}}t + c_4\right)\hat{\mathbf{j}} + c_5 \cos\left(\sqrt{\frac{k}{m}}t + c_6\right)\hat{\mathbf{k}}$$

for some constants c_1, c_2, c_3, c_4, c_5, and c_6, which is a sinusoidal wave with period

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

□

In particular, note that the position of the object is a sinusoidal wave, which implies that the object's velocity and acceleration are also sinusoidal.

Because the spring force is conservative, the total mechanical energy of a system in simple harmonic oscillation is constant.

§7.2 Damped harmonic oscillation

Prototypical example for this section: A swinging door.

A system is in **damped harmonic oscillation** if another force, such as friction, slows the motion of the system. Typically, the damping force is modeled as being proportional to and in the opposite direction of the velocity of the object in the system; this constant of proportionality is known as the **viscous damping coefficient** of the system. Symbolically,

$$\mathbf{F} = -k\mathbf{x} - c\mathbf{v}.$$

Theorem 7.2.1 (Damped harmonic oscillation)

Let

$$\zeta = \frac{c}{2\sqrt{mk}}$$

be the **damping ratio** of a damped harmonic oscillator with mass m. Then the trajectory of the system is can be modeled as

- exponentially decaying to equilibrium without oscillating if $\zeta > 1$, where larger values of ζ indicate a slower return to equilibrium,
- exponentially decaying to equilibrium as quickly as possible if $\zeta = 1$, and
- exponentially decaying with oscillation if $\zeta < 1$, where the period of oscillation is

$$2\pi\sqrt{\frac{m}{k\left(1-\zeta^2\right)}}.$$

Proof. By Newton's second law,

$$m\mathbf{a} + c\mathbf{v} + k\mathbf{x} = 0.$$

The characteristic equation of this second-order linear differential equation, $mr^2 + cr + k = 0$, has roots

$$\frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\frac{k}{m}\left(\zeta^2 - 1\right)}.$$

If $\zeta > 1$, then the roots are distinct and real, so the general solution to the differential equation is

$$\mathbf{x} = \mathbf{c}_1 e^{\left(-\frac{c}{2m} + \sqrt{\frac{k}{m}\left(\zeta^2 - 1\right)}\right)t} + \mathbf{c}_2 e^{\left(-\frac{c}{2m} - \sqrt{\frac{k}{m}\left(\zeta^2 - 1\right)}\right)t}$$

for some vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. This approximates exponential decay, and the rate of decay decreases as ζ increases.

If $\zeta = 1$, then the characteristic equation has a double root and the general solution is

$$\mathbf{x} = (\mathbf{c}_1 t + \mathbf{c}_2) e^{-\frac{c}{2m}t}$$

for some vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. Because $-\frac{c}{2m} < -\frac{c}{2m} + \sqrt{\frac{k(\zeta'^2 - 1)}{m}}$ for all $\zeta' > 1$, the decay for $\zeta = 1$ is faster than the decay for $\zeta > 1$.

Lastly, if $\zeta < 1$, then the roots are non-real, so the general solution to the differential equation is

$$\mathbf{x} = e^{-\frac{c}{2m}t}\left(c_1 \cos\left(\omega t + c_2\right)\hat{\mathbf{\imath}} + c_3 \cos\left(\omega t + c_4\right)\hat{\mathbf{\jmath}} + c_5 \cos\left(\omega t + c_6\right)\hat{\mathbf{k}}\right)$$

for some constants c_1, c_2, c_3, c_4, c_5, and c_6, where $\omega = \sqrt{\frac{k}{m}(1 - \zeta^2)}$. This is an exponentially decaying oscillation with period

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k\left(1 - \zeta^2\right)}}.$$

□

§7.3 Simple gravity pendulum

Prototypical example for this section: A grandfather clock.

A **pendulum** is a object suspended from a pivot by a string or rod of fixed length. When pushed, a pendulum oscillates because gravity exerts a restoring force on it. A **simple gravity pendulum** is one for which the object is a point particle and the rod is massless.

Theorem 7.3.1 (Simple gravity pendulum)

The period of a simple gravity pendulum with a length of l is given by

$$T \approx 2\pi\sqrt{\frac{l}{g}},$$

where g is the gravity of the planet the pendulum is on.

Proof. Let θ be the signed angle between the rod and its equilibrium position. Then the net force $\mathbf{F}$ acting on the pendulum is the component of the force $\mathbf{F}_G$ exerted by

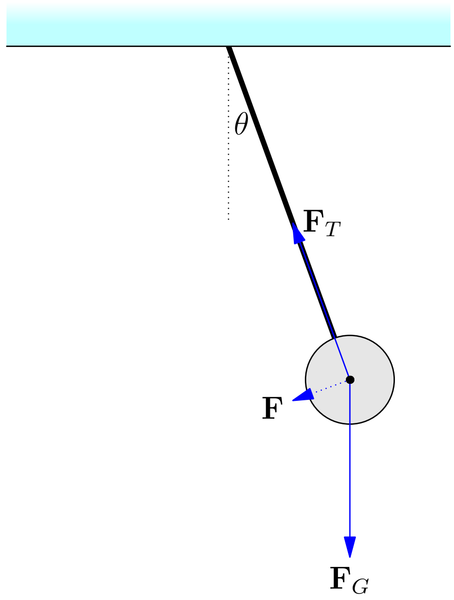

Figure 7.2: A pendulum.

gravity on the mass parallel to the direction of motion, since the tension force is directed towards the rod and the direction of motion is always perpendicular to the rod. Thus,

$$ml\left|\frac{d^2\theta}{dt^2}\right| = m\left\|\frac{d^2\mathbf{x}}{dt^2}\right\| = \|\mathbf{F}\| = mg\left|\sin\theta\right| \approx mg\left|\theta\right|$$

by the small-angle approximation, which gives the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta \approx 0,$$

since the net force on the pendulum is always in the opposite direction of its location. The characteristic equation of this second-order linear differential equation, $r^2 + \frac{g}{l} = 0$, has roots $\pm i\sqrt{\frac{g}{l}}$, so the general solution is

$$\mathbf{x} \approx \mathbf{c}_1 \cos\left(\sqrt{\frac{g}{l}}t + \mathbf{c}_2\right)$$

for some constant vectors $\mathbf{c}_1$ and $\mathbf{c}_2$, which is a sinusoidal wave with period

$$T \approx 2\pi\sqrt{\frac{l}{g}}.$$

□

Remark 7.3.2 — Due to the Taylor expansion for $\sin\theta$, the error in the period formula is of order θ^3.

§7.4 Compound pendulum

Prototypical example for this section: A swinging rod.

A **compound pendulum** is a rigid body, such as a rod with mass, free to rotate about a fixed horizontal axis ℓ.

Theorem 7.4.1 (Compound pendulum)
The period of a simple gravity pendulum with mass m is

$$T \approx 2\pi\sqrt{\frac{I_\ell}{mgr}},$$

where I_ℓ is the moment of inertia about ℓ, r is the distance from the center of mass to ℓ, and g is the gravity of the planet the pendulum is on.

Proof. By Corollary 4.7.2 and Corollary 4.4.2,

$$I_\ell \frac{d^2\theta}{d\theta^2} = \|\boldsymbol{\tau}\| = mgr\sin\theta \approx mgr\theta$$

by the small-angle approximation, which gives the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{mgr}{I_\ell}\theta \approx 0,$$

since the net force on the pendulum is always in the opposite direction of its location. The characteristic equation of this second-order linear differential equation, $r^2 + \frac{mgr}{I_\ell} = 0$, has roots $\pm i\sqrt{\frac{mgr}{I_\ell}}$, so the general solution is

$$\mathbf{x} \approx \mathbf{c}_1 \cos\left(\sqrt{\frac{mgr}{I_\ell}}t + \mathbf{c}_2\right)$$

for some constant vectors $\mathbf{c}_1$ and $\mathbf{c}_2$, which is a sinusoidal wave with period

$$T \approx 2\pi\sqrt{\frac{I_\ell}{mgr}}.$$

□

§7.5 Foucault pendulum

A **Foucault pendulum** is a pendulum designed to demonstrate the Earth's rotation. Foucault pendulums are typically found as science displays at museums and consist of a large pendulum that swings for a long period of time. The plane in which a Foucault pendulum oscillates in rotates very slowly throughout the course of a day due to the Earth's rotation.

Theorem 7.5.1 (Foucault pendulum)
The period of precession of the plane of oscillation of a Foucault pendulum is

$$T \approx \frac{d}{\sin\phi}$$

where d is the length of a sidereal day and ϕ is the latitude of the pendulum.

Proof. Work in the reference frame fixing the Earth. The only fictitious forces acting on the pendulum are the centrifugal force and the Coriolis force, and the centrifugal force can be neglected because it is constant and can be treated as effectively changing the local gravity. Additionally, assume that the Foucault pendulum swings through a small angle, so that its trajectory can be considered to be parallel to the ground.

The Coriolis force on the pendulum is

$$\mathbf{F} = 2m\mathbf{v} \times \boldsymbol{\omega},$$

where $\mathbf{v}$ is the velocity of the pendulum and $\boldsymbol{\omega}$ is the rotation rate of the Earth. Since the angle between $\mathbf{v}'$ and $\boldsymbol{\omega}$ is ϕ, the magnitude of the Coriolis force is

$$2m \|\mathbf{v}\| \|\boldsymbol{\omega}\| \sin\phi,$$

and it points perpendicularly to the velocity of the pendulum, with direction flipping depending on the direction of the swinging.

Now, the pendulum's motion can be modeled as existing at the north pole of a imaginary planet whose angular speed is $\|\boldsymbol{\omega}\| \sin\phi$, since the Coriolis force on such a planet exhibits the exact same property as the one acting on the pendulum on Earth.

Working in the inertial frame of reference containing the imaginary planet, it becomes clear that the imaginary planet rotates under the pendulum, whose plane of oscillation is fixed, once every $\frac{1}{\|\boldsymbol{\omega}\| \sin\phi}$ days. Therefore, the plane of oscillation of the Foucault pendulum rotates with a period of

$$\frac{1}{\|\boldsymbol{\omega}\| \sin\phi} = \frac{d}{\sin\phi}. \qquad \square$$

§7.6 Gyroscopic precession

Prototypical example for this section: A spinning top.

A **gyroscope** is a rapidly spinning object, typically a wheel or a spinning object, mounted such that its axis of rotation is free to change.

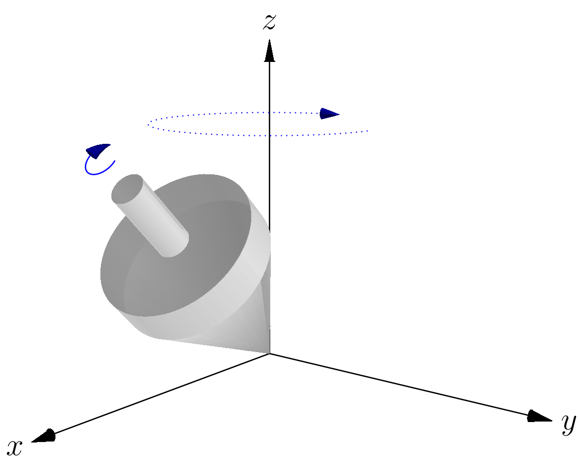

Figure 7.3: A gyroscope.

The axis of rotation of a gyroscope itself rotates; this rotation is known as **precession**.

Theorem 7.6.1 (Precession formula)
The period of precession of a gyroscope with mass m is

$$T \approx 2\pi \frac{I\omega}{mgr},$$

where I is the moment of inertia around its spinning axis, ω is the angular speed of the gyroscope around the spinning axis, and r is the distance from the center of mass of the gyroscope to the fixed point.

Proof. Let θ be the tilt of the gyroscope. The magnitude of the torque exerted on the gyroscope relative to the origin is

$$\tau = mgr \sin\theta$$

by Corollary 4.4.2, and it points in a direction parallel to the ground and perpendicular to the axis of rotation. Additionally, since $\|\boldsymbol{\omega}\|$ is much larger than the speed of precession,

$$\mathbf{L} \approx I\boldsymbol{\omega},$$

as the speed of precession can be neglected. Since the projection of angular momentum onto the plane parallel to the ground has a magnitude of approximately

$$I\omega \sin\theta$$

and torque is the derivative of angular momentum, the period of rotation is

$$T \approx 2\pi \frac{I\omega \sin\theta}{mgr\sin\theta} = 2\pi \frac{I\omega}{mgr}. \qquad \square$$

§7.7 A few harder problems to think about

Problem 7A. A massless spring hangs from the ceiling, and a mass is hung from the bottom of it. The mass is supported so that initially the tension in the spring is zero. The mass is then suddenly released. At the bottom of its trajectory, the mass is 5 centimeters from its original position. Find its oscillation period.

Problem 7B. A mass m is hanging at the end of an ideal spring attached to a ceiling. If the spring has spring constant k and the system has a period of T, find the maximum speed of the mass.

Problem 7C. A mass m is glued inside a massless hollow rod of length ℓ at an unknown location. When the rod is pivoted at one end, the period of small oscillations is T. When the rod is pivoted at the other end, the period of small oscillations is $2T$. How far is the mass from the center?

Problem 7D. A mass m is attached to a thin rod of length ℓ so that it can freely spin in a vertical circle with period T. Determine the difference in the tensions in the rod when the mass is at the top and the bottom of the circle.

Problem 7E. Three identical masses are connected with identical rigid rods and pivoted at point A, as shown in Figure 7.4. If the lowest mass receives a small horizontal push to the left, it oscillates with period T_1. If it instead receives a small push into the page, it oscillates with period T_2. Determine $\frac{T_1}{T_2}$.

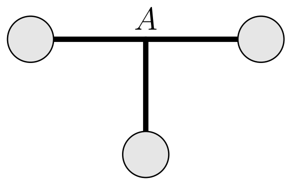

Figure 7.4: A compound pendulum with three masses.

Problem 7F. A uniform disk of mass m and radius r is attached at its edge to a flexible pivot on the ceiling. It is given a small displacement *perpendicular* to the plane of the disk so that it begins to oscillate perpendicular to the plane of the disk. What is its period of oscillation?

Problem 7G. An object of mass 1 kg is attached to a platform of mass 4 kg with a spring of spring constant 400 $\frac{\mathrm{N}}{\mathrm{m}}$, as shown in Figure 7.5. There is no friction between the object and the platform, and the coefficient of static friction between the platform and the ground is 0.1. The object is placed at its equilibrium position, and then given a horizontal velocity v. For what values of v will the platform never slip on the ground?

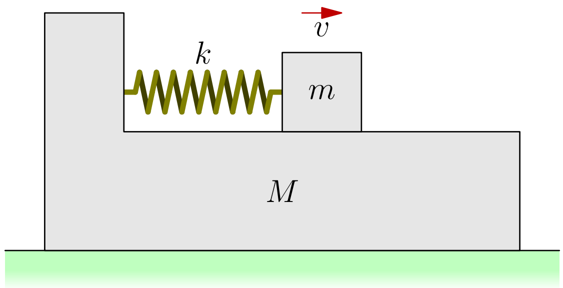

Figure 7.5: A spring on a platform.

Problem 7H. A conical pendulum of length ℓ swings in a horizontal circle of radius r, as shown in Figure 7.6. Determine the period of the pendulum.

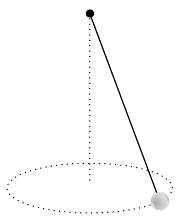

Figure 7.6: A conical pendulum.

Problem 7I. Two particles with masses m_1 and m_2 are connected by a massless rigid rod of length ℓ and placed on a horizontal frictionless table. At time $t = 0$, the first mass receives an impulse perpendicular to the rod, giving it speed v. At this moment, the second mass is at rest. Determine the next time the second mass is at rest.

Problem 7J. A mass M sits on top of a vertical spring of spring constant k, in equilibrium. A mass m is held a height h above it. The mass M is held a height h above it. The mass M is then pushed downward by a distance x, and both masses are released from rest simultaneously.

- For what value of h will the two masses first collide when M first returns to its equilibrium position?
- Assume that h takes the value found in the previous question, and that the collision between the two masses is perfectly elastic. For what value of x will m rebound to a maximum height that is exactly equal to its original height?

Problem 7K. A rod passes perpendicularly to a solid, uniform disk of mass m and radius r, and the disk is pushed by a small amount so that the disk starts oscillating. Across all possible rod locations, what is the minimum period of the resulting pendulum?

Problem 7L. A simple pendulum has a length of l. If its mass is initially at the bottommost position and is given a velocity of $\sqrt{4lg}$, will the pendulum ever reach the uppermost position?

8 Celestial mechanics

Celestial mechanics is the branch of physics that deals with the motion of **celestial bodies** in outer space, which are idealized as spheres with uniform density.

§8.1 Gravitational potential energy

Prototypical example for this section: An asteroid near Earth.

Gravity is conservative because the total work needed to move a particle from one point to another point in the presence of a fixed mass located at the origin is independent of the path taken, since the gravitational force is spherically symmetric and hence only the distances from $\mathbf{r}_1$ and $\mathbf{r}_2$ to the origin matter; in particular, no work is gained from motion perpendicular to the position vector. Thus, given an object with fixed mass, a potential at each point can be assigned, known as the **gravitational potential energy**, measured in joules.

Remark 8.1.1 — This value is different from the gravitational potential energy of an object near Earth from Section 6.3. However, it should be clear from context which gravitational potential energy to consider.

By convention, the gravitational potential energy between two particles that are infinitely far away is zero. Therefore, the gravitational potential energy of a system can be thought of as the amount of work necessary to assemble its constituent parts, given that the particles start infinitely far away from each other. In particular, the gravitational potential energy of two particles m_1 and m_2 separated a distance r from each other is

$$U_g = \int_\infty^r \frac{Gm_1m_2}{r'^2}dr' = -Gm_1m_2\int_r^\infty \frac{1}{r'^2}dr' = -\frac{Gm_1m_2}{r}.$$

§8.2 Shell theorem

The **shell theorem** explains why the gravity of planets can be treated as originating from a single point at its center.

Theorem 8.2.1 (Shell theorem for external objects)
A spherically symmetric body affects external objects gravitationally as if all of its mass were concentrated at its center.

Proof. It suffices to assume that the spherically symmetric body is a uniform shell by decomposing it into a union of disjoint, infinitesimally thin shells. Additionally, assume that the shell has radius R, mass m_1, and is centered at the origin. Lastly, assume that the external object has mass m_2 and lies on the positive x-axis at a distance r away from the origin.

By Archimedes' hat-box theorem, the mass of a vertical ring at with width dx is $\frac{dx}{2r'}m_1$. By the Pythagorean Theorem, the distance between the ring and the external object is

$$\sqrt{r^2 + r'^2 - 2rx},$$

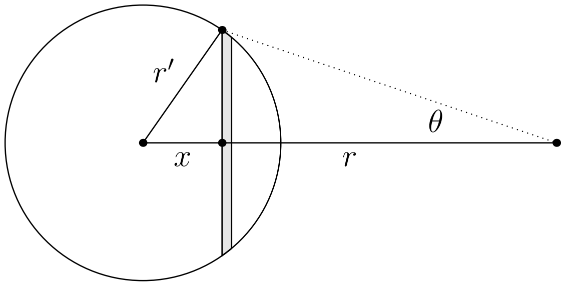

Figure 8.1: Proving the shell theorem, exterior case.

so the magnitude of the gravitational force exerted by an infinitesimal mass dm is

$$\frac{Gm_2 dm}{r^2 + r'^2 - 2rx}.$$

However, only the x-component of the force needs to be considered, because the component parallel to the yz-plane cancels out due to the symmetry of the ring. Thus, the gravitational force exerted by the ring on the external object is

$$\frac{G\frac{dx}{2r'}m_1 m_2}{r^2 + r'^2 - 2rx}\cos\theta = \frac{Gm_1 m_2 (r-x)}{2r'\left(r^2 + r'^2 - 2rx\right)^{\frac{3}{2}}}dx = \frac{Gm_1 m_2}{2r'^2} \cdot \frac{c-u}{(1-2cu+c^2)^{\frac{3}{2}}}du,$$

where $u = \frac{x}{r'}$ and $c = \frac{r}{r'} > 1$ for brevity. Since

$$\int \frac{c-u}{(1-2cu+c^2)^{\frac{3}{2}}}du = \frac{cu-1}{c^2\sqrt{1-2cu+c^2}} + C,$$

integrating force over all x between $-r'$ and r' – or equivalently, over all u between -1 and 1 – gives

$$\begin{aligned} \|\mathbf{F}_g\| &= \frac{Gm_1 m_2}{2r'^2}\int_{-1}^{1} \frac{c-u}{(1-2cu+c^2)^{\frac{3}{2}}}du \\ &= \frac{Gm_1 m_2}{2r'^2}\left(\frac{c-1}{c^2\sqrt{1-2c+c^2}} - \frac{-c-1}{c^2\sqrt{1+2c+c^2}}\right) \\ &= \frac{Gm_1 m_2}{r^2} \end{aligned}$$

using the fact that $c > 1$, which is the gravitational force as if the body's mass were concentrated as its center. □

Theorem 8.2.2 (Shell theorem for internal objects)
No net gravitational force is exerted by a uniform spherical shell on any object inside.

Proof. Let P be a point particle inside the shell, and partition the sphere into infinitesimally small regions dA. For each region dA, let dA' be the region obtained by projecting dA through P onto the other side of the shell; it suffices to show that the combined net force exerted by dA and dA' on P is zero.

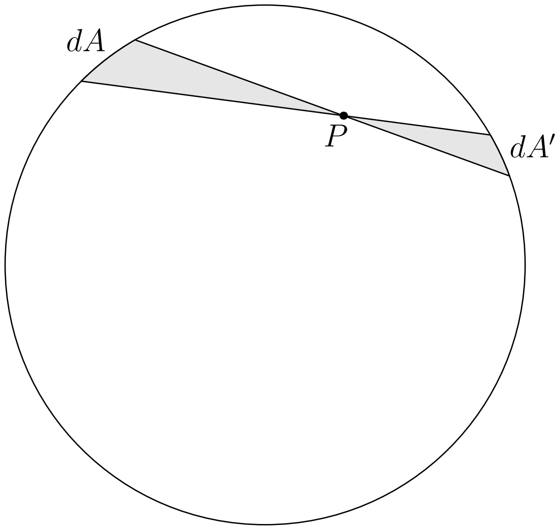

Figure 8.2: Proving the shell theorem, interior case.

Since the cone with base dA and apex P is similar to the cone with base dA' and apex P, the ratio of the areas of dA and dA' is equal to the square of the ratio of the distances from P to dA and dA'. Since gravitational force is proportional to mass and inversely proportional to the square of the distance, the force exerted by dA on P is equal in magnitude and opposite in direction to the force exerted by dA' on P, as desired. □

Corollary 8.2.3 (Gravitational force inside planet)

Inside an ideal celestial body, the gravitational force on an object is proportional to its distance from the center.

Proof. Let r be the distance from the object to the center of the celestial body. By the shell theorem, only the mass within r of the center contributes to the gravitational force. Since gravitational force is inverse quadratic with distance but volume varies cubically, the gravitational force varies linearly with r. □

§8.3 Circular orbits

Suppose a planet with mass m is orbiting a star with mass M in a circle with radius r. Then for all points on the orbit,

$$\frac{GMm}{r^2} = \|\mathbf{F}\| = \frac{mv^2}{r},$$

by the centripetal acceleration formula. Solving for v gives

$$v = \sqrt{\frac{GM}{r}}.$$

Thus, the orbital period of a planet in a circular orbit is the ratio of the circle's circumference to its speed, which is

$$T = \frac{2\pi r}{\frac{\sqrt{GM}}{r}} = 2\pi\sqrt{\frac{r^3}{GM}}.$$

The **virial theorem** gives a relationship between a planet's kinetic energy and its gravitational potential energy in circular motion.

Theorem 8.3.1 (Virial theorem)
The kinetic energy of a planet in a circular orbit is half the magnitude of its gravitational potential energy; that is,

$$U_k = -\frac{1}{2}U_g.$$

Proof. Let r be the radius of the planet's orbit, and let m be the planet's mass. Then by the kinetic energy formula,

$$U_k = \frac{1}{2}mv^2 = \frac{1}{2}\cdot\frac{GMm}{r} = -\frac{1}{2}U_g,$$

which is half of the gravitational potential energy of the planet. □

§8.4 Kepler's second law of planetary motion

Kepler's second law informally states that a planet sweeps out equal areas in equal times.

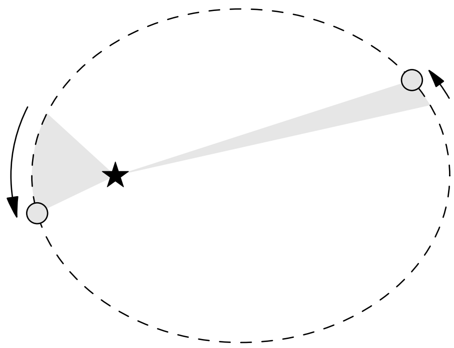

Figure 8.3: Kepler's second law.

Theorem 8.4.1 (Kepler's second law)
The line segment connecting a planet and the star it orbits sweeps out an area proportional to the duration of the time interval.

Proof. Since the mass of the star M is much larger than the mass of the planet m, the Sun's position can be assumed to be constant and centered at the origin. The area swept out in an infinitesimal time period dt is

$$\frac{1}{2}\left\|\mathbf{r}\times\mathbf{v}\right\|dt = \frac{1}{2}\left\|\frac{\mathbf{L}}{m}\right\|dt,$$

so integrating over a time period t gives the area as

$$\frac{\|\mathbf{L}\|}{2m}t.$$

Since the net force on the planet is directed towards the star, no net torque is exerted on the planet. $\mathbf{L}$ is constant by conservation of angular momentum, so area is proportional to time. □

Remark 8.4.2 — Note that the inverse-square property of gravity is never used; only the fact that no net torque is exerted is.

§8.5 Kepler's first law of planetary motion

Kepler's first law states that the shapes of all orbits are ellipses.

Theorem 8.5.1 (Kepler's first law)

The trajectory of a planet is an ellipse with the star it orbits at one of its foci.

Proof. Since the mass of the star M is much larger than the mass of the planet m, the Sun's position can be assumed to be constant and centered at the origin. By Newton's law of universal gravitation,

$$\mathbf{a} = \frac{\mathbf{F}}{m} = -\frac{GM}{\|\mathbf{x}\|^2}\frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

Since angular momentum is constant by Kepler's second law,

$$\begin{aligned}
\frac{d}{dt}[\mathbf{v}\times\mathbf{L}] &= \mathbf{a}\times\mathbf{L} \\
&= -\frac{GM}{\|\mathbf{x}\|^2}\frac{\mathbf{x}}{\|\mathbf{x}\|}\times(\mathbf{x}\times m\mathbf{v}) \\
&= -GMm\frac{(\mathbf{x}\cdot\mathbf{v})\mathbf{x}-(\mathbf{x}\cdot\mathbf{x})\mathbf{v}}{\|\mathbf{x}\|^3} \\
&= GMm\frac{\left(\sqrt{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{v}-\frac{\mathbf{x}}{\sqrt{\mathbf{x}\cdot\mathbf{x}}}\mathbf{x}\cdot\mathbf{v}}{\mathbf{x}\cdot\mathbf{x}} \\
&= GMm\frac{d}{dt}\left[\frac{\mathbf{x}}{\sqrt{\mathbf{x}\cdot\mathbf{x}}}\right].
\end{aligned}$$

Therefore,

$$\mathbf{v}\times\mathbf{L} = GMm\frac{\mathbf{x}}{\|\mathbf{x}\|}+\mathbf{c}$$

for some constant vector $\mathbf{c}$. Finally,

$$\begin{aligned}
\|\mathbf{L}\|^2 = \mathbf{L}\cdot\mathbf{L} = m(\mathbf{x}\times\mathbf{v})\cdot\mathbf{L} = m\mathbf{x}\cdot(\mathbf{v}\times\mathbf{L}) &\\
= m\mathbf{x}\cdot\left(GMm\frac{\mathbf{x}}{\|\mathbf{x}\|}+\mathbf{c}\right)&\\
= GMm^2\|\mathbf{x}\| + m\|\mathbf{x}\|\|\mathbf{c}\|\cos\theta,&
\end{aligned}$$

where θ is the angle between $\mathbf{x}$ and the constant vector $\mathbf{c}$. Solving for $\|\mathbf{x}\|$ gives

$$\|\mathbf{x}\| = \frac{\|\mathbf{L}\|^2}{GMm^2+m\|\mathbf{c}\|\cos\theta} = \frac{\frac{\|\mathbf{L}\|^2}{GMm^2}}{1+\frac{\|\mathbf{c}\|}{GMm}\cos\theta},$$

which is the polar equation of a conic with eccentricity $\frac{\|\mathbf{c}\|}{GMm}$ and semi-latus rectum $\frac{\|\mathbf{L}\|^2}{GMm^2}$. If the planet does indeed orbit, the conic must be an ellipse. □

The point for which the orbit of a planet around the Sun is closest to the Sun is the planet's **perihelion**, and the point for which the orbit is farthest from the Sun is the planet's **aphelion**. Similarly, the point for which the orbit of a satellite around the Earth is closest to the Earth is the satellite's **perigee**, and the point for which the orbit is farthest from the Earth is the satellite's **apogee**. In general, such extreme points are referred to as **apsides**.

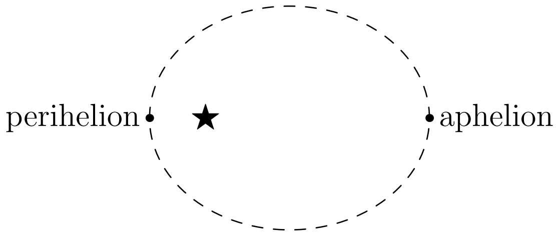

Figure 8.4: A planet's perihelion and aphelion.

§8.6 Kepler's third law of planetary motion

Kepler's third law gives a relationship between a planet's period and the semi-major axis of its orbit.

Theorem 8.6.1 (Kepler's third law)
The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

Proof. Let a be the semimajor axis of the ellipse guaranteed by Kepler's first law. The semi-latus rectum of the ellipse is $\frac{||\mathbf{L}||^2}{GMm^2}$ by the proof of Kepler's first law, so it satisfies

$$(2ae)^2 + \left(\frac{||\mathbf{L}||^2}{GMm^2}\right)^2 = \left(2a - \frac{||\mathbf{L}||^2}{GMm^2}\right)^2$$

by the Pythagorean theorem, where e is the eccentricity of the ellipse.

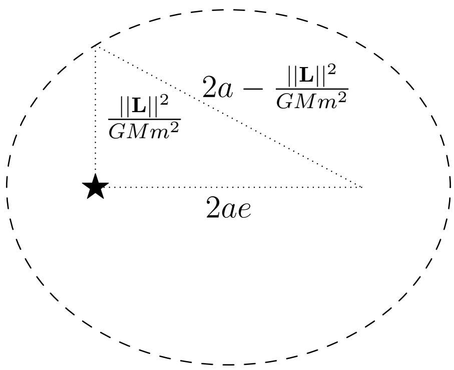

Figure 8.5: Proving Kepler's third law.

Simplifying gives

$$1 - e^2 = \frac{\|\mathbf{L}\|^2}{GMm^2a}.$$

Lastly, Kepler's second law gives

$$\frac{\|\mathbf{L}\|}{2m}T = \pi a^2\sqrt{1-e^2} = \pi a^2\sqrt{\frac{\|\mathbf{L}\|^2}{GMm^2a}} = \frac{\pi\,\|\mathbf{L}\|}{m\sqrt{GM}}a^{\frac{3}{2}},$$

using the ellipse area formula, so

$$T^2 = \frac{4\pi^2}{GM}a^3. \qquad \square$$

§8.7 Two-body problem

The **two-body problem** is the problem of determining the trajectories of two celestial bodies given that they only interact through gravity. It turns out that both celestial bodies will end up orbiting their center of mass elliptically.

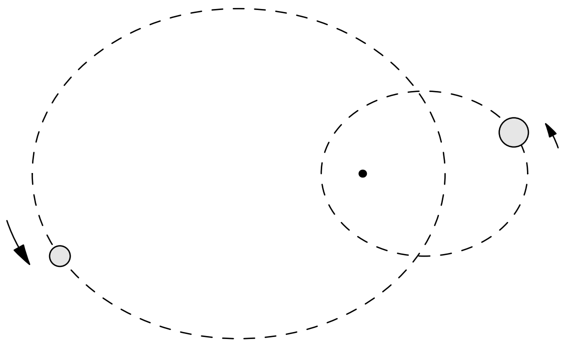

Figure 8.6: Two celestial bodies orbiting each other.

To prove this, consider two celestial bodies with masses m_1 and m_2 that orbit each other with trajectories $\mathbf{x}_1$ and $\mathbf{x}_2$. By Newton's second and third laws,

$$\frac{d^2}{dx^2}\left[\mathbf{x}_2 - \mathbf{x}_1\right] = \frac{\mathbf{F}_2}{m_2} - \frac{\mathbf{F}_1}{m_1} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_2 = -\frac{\mathbf{x}_2 - \mathbf{x}_1}{\|\mathbf{x}_2 - \mathbf{x}_1\|}\frac{G(m_1+m_2)}{\|\mathbf{x}_2 - \mathbf{x}_1\|^2}$$

so the displacement between the planets satisfies the differential equation analogous to the one that sets up the proof of Kepler's first law. Hence, the displacement vector, by Kepler's first law, traces out an ellipse.

By conservation of momentum, the center of mass of the two celestial bodies moves at a constant velocity. Since the masses of the two bodies are constant, their center of mass is always at a constant location relative to their displacement vector. By homothety, this implies that both bodies orbit elliptically around their center of mass.

Remark 8.7.1 — In particular, note that the orbits, relative to their center of mass, are confined to a plane.

§8.8 A few harder problems to think about

Problem 8A. A satellite is in a circular orbit around the Earth. Over a long period of time, the effects of air resistance decrease the satellite's total energy by 1 J. How does the kinetic energy of the satellite change?

Problem 8B. A planet is orbiting a star in a circular orbit of radius r. Over a very long period of time, much greater than the period of the orbit, the star slowly and steadily loses 1% of its mass. Throughout the process, the planet's orbit remains approximately circular. The final orbit radius is kr for some real number k. Determine k to the nearest hundredth.

Problem 8C. A satellite is following an elliptical orbit around the Earth, as shown in 8.7. Its engines are capable of providing a one-time impulse of a fixed magnitude. To maximize the energy of the satellite, where and in what direction should the impulse be applied?

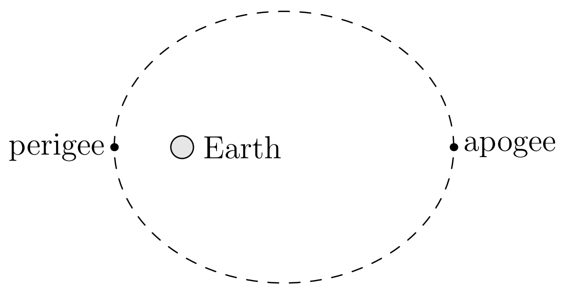

Figure 8.7: A satellite's orbit around the Earth.

Problem 8D. Two planets A and B has masses $m_A = 2m_B$. They orbit a star in circular orbits of radii $r_A = 3r_B$. Compair the planets' kinetic energies, as well as their angular momenta.

Problem 8E. A very long cylinder of dust is spinning about its axis with constant angular velocity. Let r be the distance from the axis. If the dust is only held together by gravity, the density of the dust is proportional to r^k for some integer k. Determine k.

Problem 8F. Two satellites are initially in identical circular orbits around the Sun, with orbital speed $1 \times 10^4 \, \frac{\text{m}}{\text{s}}$. The first satellite fires its thrusters toward the Sun, and quickly obtains a radial velocity of $1 \, \frac{\text{m}}{\text{s}}$. The second satellite instead fires its thrusters behind it, and quickly increases its tangential velocity by $\Delta\mathbf{v}$. If the two satellites subsequently perform orbits with the same period, determine $\|\Delta\mathbf{v}\|$ to one significant digit.

Problem 8G. A spherical cloud of dust has uniform mass density ρ and radius R. Satellite A of negligible mass is orbiting the cloud at its edge, in a circular orbit of radius R, and satellite B is orbiting the cloud just inside the cloud, in a circular orbit of radius r, with $r < R$. Compare the periods of A and B, as well as their speeds.

Problem 8H. A cavity of radius $\frac{r}{2}$ is dug out of a spherical planet with uniform mass density of mass m and radius r. What is the magnitude of the gravitational field at point P in Figure 8.8?

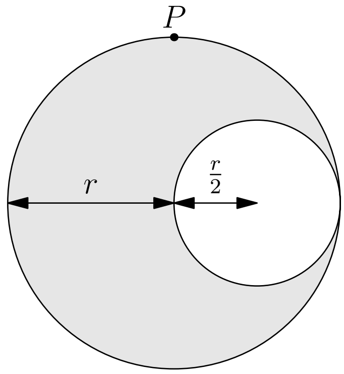

Figure 8.8: A planet with a cavity.

Problem 8I. A particle of mass m is placed at the center of a hemispherical shell of radius r and mass density σ, which has units $\frac{\text{kg}}{\text{m}^2}$, as shown in Figure 8.9. Determine the gravitational force of the shell on the particle.

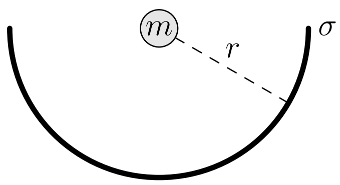

Figure 8.9: A hemispherical shell.

Problem 8J. Spaceman Fred's trusty pellet sprayer is held at rest a distance h away from the center of Planet Orb, which has radius r much less than h. The pellet sprayer ejects pellets radially outward, uniformly in the plane of the page, as shown in Figure 8.10. These pellets are all launched with the same speed v, so that a pellet launched directly away from Orb by the pellet sprayer can just barely escape it. What fraction of the pellets eventually lands on Orb? You may use the small angle approximation.

Problem 8K. Two satellites are in circular orbits around a star with equal radius r, speed v, and period T. The satellites are initially diametrically opposite each other. In order to meet the second satellite in time $\frac{1}{2}T$, the first satellite should decrease its speed to kv, for some real constant k. Find k.

Problem 8L. Consider two identical masses that interact only by gravitational attraction to each other. If one mass is fixed in place and the other is released from rest, then the two masses collide in time T. Determine the time it takes for them to collide if both masses are released from rest.

Problem 8M. A straight tunnel is dug between two cities on Earth's surface, and a frictionless train rolls from one city to the other city. Prove that the duration of the ride does not depend on the choice of cities.

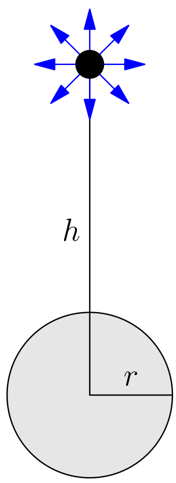

Figure 8.10: A pellet sprayer.

II

Appendix

Part II: Contents

A Glossary of notations

This is a list of conventions used in this document.

§A.1 General

By convention, boldface variables represent vectors and non-boldface variables represent scalars. All axioms of physics are defined as laws in the text, and all derived facts are listed as theorems, corollaries, or propositions. In particular, results are listed in the order of significance

$$\text{law} > \text{theorem} > \text{corollary} > \text{proposition}.$$

§A.2 Mathematics

- V: vector space
- $\mathbb{R}$: real numbers
- $\|\bullet\|$: magnitude
- $\cdot$: dot product
- $\times$: cross product
- proj: projection
- $\bullet^\top$: transpose
- $[\bullet]_\times$: matrix cross product operator
- $\hat{\mathbf{i}}$: unit vector in the x-direction
- $\hat{\mathbf{j}}$: unit vector in the y-direction
- $\hat{\mathbf{k}}$: unit vector in the z-direction
- $\hat{\mathbf{n}}$: generic unit vector
- $\mathbf{0}$: zero vector
- λ: eigenvalue
- θ: polar angle
- φ: azimuthal angle
- r: radius
- s: radius
- i: square root of -1
- c: constant

- C: constant
- $\mathbf{c}$: constant
- A: area
- P: point
- e: Euler's constant
- e: eccentricity

§A.3 Units

Abbreviation for units is as follows:

- m: meter
- s: second
- kg: kilogram
- $\mathrm{N} = \frac{\mathrm{kg \cdot m}}{\mathrm{s}^2}$: newton
- $\mathrm{J} = \frac{\mathrm{kg \cdot m}^2}{\mathrm{s}^2}$: joule
- $\mathrm{W} = \frac{\mathrm{kg \cdot m}^2}{\mathrm{s}^3}$: watt
- C: coulomb

§A.4 Diagrams

Diagram conventions for classical mechanics are as follows.

- The ground is light green
- The ceiling is cyan
- Generic objects are light gray
- Auxiliary objects are orange
- Generic solids are white
- Force vectors are dark blue arrows
- Velocity vectors are black arrows
- Ramps are light blue triangles
- Walls are dark red
- Ideal strings are red line segments
- Rods are black line segments
- Pulleys are light pink circles
- Springs are olive zigzags

§A.5 Classical mechanics

Kinematics notation is as follows.

- t: time (s)
- $\mathbf{x}$: position (m)
- $\mathbf{v}$: velocity $\left(\frac{\mathrm{m}}{\mathrm{s}}\right)$
- v: speed $\left(\frac{\mathrm{m}}{\mathrm{s}}\right)$
- $\mathbf{a}$: acceleration $\left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)$
- $\mathbf{a}_T$: tangential acceleration $\left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)$
- $\mathbf{a}_C$: centripetal acceleration $\left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)$
- ω: angular speed $\left(\frac{\mathrm{rad}}{\mathrm{s}} = \frac{1}{\mathrm{s}}\right)$
- r: radius (m)
- θ: angular displacement (rad = unitless)
- α: angular acceleration $\left(\frac{\mathrm{rad}}{\mathrm{s}^2} = \frac{1}{\mathrm{s}^2}\right)$
- T: period (s)
- $\boldsymbol{\omega}$: angular velocity $\left(\frac{\mathrm{rad}}{\mathrm{s}} = \frac{1}{\mathrm{s}}\right)$
- $\boldsymbol{\alpha}$: angular acceleration $\left(\frac{\mathrm{rad}}{\mathrm{s}^2} = \frac{1}{\mathrm{s}^2}\right)$
- m: mass (kg)

Force notation is as follows.

- $\mathbf{F}$: force $\left(\mathrm{N} = \frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}\right)$
- G: gravitational constant $\left(\approx 6.674\,\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{kg}^2} = 6.674 \times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}\right)$
- g gravity of Earth $\left(\approx 9.807\,\frac{\mathrm{N}}{\mathrm{m}} = 9.807\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)$
- $\mathbf{F}_g$: gravitational force $\left(\mathrm{N} = \frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}\right)$
- $\mathbf{F}_N$: normal force $\left(\mathrm{N} = \frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}\right)$
- $\mathbf{F}_f$: frictional force $\left(\mathrm{N} = \frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}\right)$
- μ_{k}: coefficient of kinetic friction (unitless)
- μ_{s}: coefficient of static friction (unitless)
- $\mathbf{F}_T$: tension force $\left(\mathrm{N} = \frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}\right)$
- $\mathbf{F}_s$: spring force $\left(\mathrm{N} = \frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}\right)$

- k: spring constant $\left(\frac{\text{N}}{\text{m}} = \frac{\text{kg}}{\text{s}^2}\right)$
- $\mathbf{F}_D$: drag force $\left(\text{N} = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}\right)$
- ρ: density $\left(\frac{\text{kg}}{\text{m}^3}\right)$
- A: area (m^2)
- C_D: drag coefficient (unitless)

Linear dynamics notation is as follows.

- $\mathbf{x}_{\text{cm}}$: position of center of mass (m)
- $\mathbf{v}_{\text{cm}}$: velocity of center of mass $\left(\frac{\text{m}}{\text{s}}\right)$
- $\mathbf{a}_{\text{cm}}$: acceleration of center of mass $\left(\frac{\text{m}}{\text{s}^2}\right)$
- $\mathbf{p}$: momentum $\left(\frac{\text{kg}\cdot\text{m}}{\text{s}}\right)$
- $\mathbf{J}$: impulse $\left(\frac{\text{kg}\cdot\text{m}}{\text{s}}\right)$
- v_t: terminal velocity $\left(\frac{\text{m}}{\text{s}}\right)$

Rotational dynamics notation is as follows.

- ℓ: axis
- r: distance to axis (m)
- I_ℓ: moment of inertia about ℓ $(\text{kg}\cdot\text{m}^2)$
- l: length (m)
- $\mathcal{P}$: plane
- $\mathbf{I}$: moment of inertia tensor $(\text{kg}\cdot\text{m}^2)$
- $\mathbf{I}_{\text{cm}}$: moment of inertia tensor with respect to center of mass $(\text{kg}\cdot\text{m}^2)$
- $\mathbf{x}_0$: position (m)
- $\boldsymbol{\tau}$: torque $\left(\text{N}\cdot\text{m} = \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}\right)$
- $\mathbf{x}'$: relative position (m)
- $\mathbf{p}'$: relative momentum $\left(\frac{\text{kg}\cdot\text{m}}{\text{s}}\right)$
- $\mathbf{L}$: angular momentum $\left(\frac{\text{kg}\cdot\text{m}^2}{\text{s}}\right)$
- $\mathbf{L}_s$: spin angular momentum $\left(\frac{\text{kg}\cdot\text{m}^2}{\text{s}}\right)$
- $\mathbf{I}_{\text{body}}$: moment of inertia tensor in a reference frame fixing the rigid body $(\text{kg}\cdot\text{m}^2)$
- $\mathbf{r}$: position relative to a point on an axis (m)

Energy notation is as follows.

- W: work $\left(\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}\right)$
- U: potential energy $\left(\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}\right)$
- U_g: gravitational potential energy $\left(\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}\right)$
- h: height (m)
- U_s: spring potential energy $\left(\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}\right)$
- E_k: kinetic energy $\left(\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}\right)$
- E_t: translational kinetic energy $\left(\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}\right)$
- E_r: rotational kinetic energy $\left(\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}\right)$
- P: power $\left(\mathrm{W} = \frac{\mathrm{J}}{\mathrm{s}} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^3}}\right)$

Oscillation notation is as follows.

- c: viscous damping coefficient $\left(\frac{\frac{\mathrm{N}}{\mathrm{m}}}{\mathrm{s}} = \frac{\mathrm{kg}}{\mathrm{s}}\right)$
- ζ: damping ratio (unitless)
- ϕ: latitude (rad = unitless)

Celestial mechanics notation is as follows.

- U_g: gravitational potential energy $\left(\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}\right)$