Vieta's Formulae

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Vieta's formulae are formulae that relate the coefficients of a single-variable polynomial to certain sums of products of its roots. They can be used to evaluate symmetric expressions involving the roots of a given polynomial.

Vieta's Formulae. Suppose the polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

has roots $r_1, r_2, \ldots, r_n$, counted with multiplicity. Then

$$\sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \prod_{j=1}^{k} r_{i_j} = (-1)^k \frac{a_{n-k}}{a_n}$$

for every integer k between 1 and n, inclusive. That is,

$$\begin{aligned} r_1 + r_2 + \ldots + r_{n-1} + r_n &= -\frac{a_{n-1}}{a_n} \\ r_1 r_2 + r_1 r_3 + \ldots + r_{n-1} r_n &= \frac{a_{n-2}}{a_n} \\ &\vdots \\ r_1 r_2 \ldots r_n &= (-1)^n \frac{a_0}{a_n}. \end{aligned}$$

Vieta's Formulae for Quadratics. Suppose the quadratic $a^2 + bx + c$ has roots r_1 and r_2. Then

$$r_1 + r_2 = \frac{-b}{a}$$

and

$$r_1 r_2 = \frac{c}{a}.$$

Vieta's Formulae for Cubics. Suppose the cubic $ax^3 + bx^2 + cx + d$ has roots r_1, r_2, and r_3. Then

$$r_1 + r_2 + r_3 = \frac{-b}{a}$$

$$r_1 r_2 + r_1 r_3 + r_2 r_3 = \frac{c}{a}$$

$$r_1 r_2 r_3 = \frac{-d}{a}.$$

Example 1. If r and s are the roots of the quadratic $x^2 - 4x + 9$, find $r + s$, rs, and $r^2 + s^2$.

Example 2. Let r, s, and t be the roots of $x^3 - 6x^2 + 5x - 7$. Find

$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2}.$$

Example 3. The roots r_1, r_2, and r_3 of $x^3 - 2x^2 - 11x + a$ satisfy

$$r_1 + 2r_2 + 3r_3 = 0.$$

Find all possible values of a.

Example 4. Let r, s, and t be the roots of $x^3 - 3x^2 + 1$.

- Find a polynomial whose roots are $r + 3$, $s + 3$, $t + 3$.

- Find a polynomial whose roots are $\frac{1}{r+3}$, $\frac{1}{s+3}$, and $\frac{1}{t+3}$.

- Compute
$$\frac{1}{r+3} + \frac{1}{s+3} + \frac{1}{t+3}.$$

Example 5. If the polynomial $x^4 + 3x^3 + 11x^2 + 9x + A$ has roots k, l, m, n such that $kl = mn$, find A.

Problems

Problem 1. Let r and s be the roots of $x^2 + 3x + 1$. Find

$$\frac{r}{s} + \frac{s}{r}.$$

Problem 2. Let r, s, and t be the roots of $5x^3 - 11x^2 + 7x + 3$. Evaluate

$$r(1 + s + t) + s(1 + t + r) + t(1 + r + s).$$

Problem 3. The sum of the reciprocals of the roots of $3x^2 + 7x + k$ is $\frac{7}{3}$. Find k.

Problem 4. Suppose r, s, and t are the roots of $x^3 - 2x^2 + 3x - 4$. Find

$$(r+1)(s+1)(t+1).$$

Problem 5. Three of the roots of $x^4 + ax^2 + bx + c$ are 2, -3, and 5. Find the value of $a + b + c$.

Problem 6. If the roots of $3x^3 - 14x^2 + x + 62$ are r, s, and t, determine

$$\frac{1}{r+4} + \frac{1}{s+4} + \frac{1}{t+4}.$$

Problem 7. The solutions of $x^3 - 3x^2 + kx + 15 = 0$ form an arithmetic progression. Find k.

Problem 8. Let a and b be the roots of $x^2 - mx + 2$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. Find q.

Problem 9. Let a and b be complex numbers satisfying

$$a^2 + b^2 = 5$$
$$a^3 + b^3 = 7.$$

Find the maximum possible value of $a + b$.

Problem 10. Suppose that the roots of $x^3 + 3x^2 + 4x - 11$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$. Find t.

Problem 11. For nonzero constants c and d, the equation

$$4x^3 - 12x^2 + cx + d$$

has two solutions which add to zero. Find $\frac{d}{c}$.

Problem 12. Let x_1 and x_2 be the roots of $x^2 + 3x + 1 = 0$. Compute

$$\left(\frac{x_1}{x_2 + 1}\right)^2 + \left(\frac{x_2}{x_1 + 1}\right)^2.$$

Problem 13. Let x, y, and z be three distinct complex numbers satisfying

$$x^3 + 5y + 5z = y^3 + 5x + 5z = z^3 + 5x + 5y = 5.$$

What is $x^2 + y^2 + z^2$?

Problem 14. Given that

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 + z^3 &= 7, \end{aligned}$$

find $x^5 + y^5 + z^5$.

Problem 15. Determine all real numbers a for which

$$16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$$

has exactly four distinct real roots that form a geometric progression.

Challenge Problems

Challenge 1. Determine all solutions (x, y, z) to

$$\begin{aligned} x + y + z &= 3 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 + z^3 &= 3. \end{aligned}$$

Challenge 2. The product of two of the four roots of

$$x^4 - 18x^3 + kx^2 + 200x - 1984$$

is -32. Find k.

Challenge 3. The real numbers w, a, b, c are distinct, such that there exist real numbers x, y, and z satisfying

$$\begin{aligned} x + y + z &= 1 \\ xa^2 + yb^2 + zc^2 &= w^2 \\ xa^3 + yb^3 + zc^3 &= w^3 \\ xa^4 + yb^4 + zc^4 &= w^4. \end{aligned}$$

Express w in terms of a, b, c.

Challenge 4. Find all polynomials whose coefficients are either -1 or 1 and have all real roots.

Challenge 5. Let a and b be positive integers such that

$$\frac{a^2 + b^2}{ab + 1}$$

is an integer. Prove that it is a square number.