Principle of Inclusion-Exclusion

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The *principle of inclusion-exclusion* is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

The main difficulty in solving combinatorics is often recognizing when to apply the principle of inclusion-exclusion; the execution of the principle is generally not difficult but may be tedious at times. It helps to draw Venn diagrams when applying the principle of inclusion-exclusion for two sets or three sets.

Two-set Principle of Inclusion-Exclusion. Let A and B be finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Three-set Principle of Inclusion-Exclusion. Let A, B, and C be finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Principle of Inclusion-Exclusion. Let $S_1, S_2, \ldots, S_n$ be finite sets. Then

$$\left|\bigcup_i A_i\right| = \sum_i |S_i| - \sum_{i<j} |S_i \cap S_j| + \cdots + (-1)^{n-1} |S_1 \cap \cdots \cap S_n| .$$

Example 1. How many integers in the set $\{1, 2, \ldots, 100\}$ are not divisible by 2, 3 or 5?

Example 2. Determine the number of arrangements of "CATCATCAT" that contain the substring "CAT".

Example 3. Ang, Ben, and Jasmin each have 5 blocks, colored red, blue, yellow, white, and green; and there are 5 empty boxes. Each of the people randomly and independently of the other two people places one of their blocks into each box. Find the probability that at least one box receives 3 blocks of the same color.

Example 4. Six people of different heights are getting in line to buy donuts. Compute the number of ways they can arrange themselves in line such that no three consecutive people are in increasing order of height, from front to back.

1 Problems

Problem 1. There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

Problem 2. What is the sum of all integers from 1 to 100 that are multiples of 2 or 3?

Problem 3. Call a number prime-looking if it is composite but not divisible by $2, 3$, or 5. The three smallest prime-looking numbers are $49, 77$, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

Problem 4. Find the number of positive integers that are divisors of at least one of 10^{10}, 15^7, 18^{11}.

Problem 5. Determine the number of ways to seat four couples in a row so that nobody sits next to their spouse.

Problem 6. Brian has 5 uniquely colored fish but only 3 identical fish tanks. If each fish tank must have at least one fish, how many different ways are there for Brian to put the fish in tanks?

Problem 7. Each unit square of a 3-by-3 unit-square grid is to be colored either blue or red. For each square, either color is equally likely to be used. Find the probability of obtaining a grid that does not have a 2-by-2 red square.

Problem 8. A 7×1 board is completely covered by $m \times 1$ tiles without overlap; each tile may cover any number of consecutive squares, and each tile lies completely on the board. Each tile is either red, blue, or green. Find the number of tilings of the 7×1 board in which all three colors are used at least once.

Problem 9. While watching a show, Ayako, Billy, Carlos, Dahlia, Ehuang, and Frank sat in that order in a row of six chairs. During the break, they went to the kitchen for a snack. When they came back, they sat on those six chairs in such a way that if two of them sat next to each other before the break, then they did not sit next to each other after the break. Find the number of possible seating orders they could have chosen after the break.

Problem 10. Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Find the probability that each delegate sits next to at least one delegate from another country.

Problem 11. Call a set S product-free if there do not exist $a, b, c \in S$ (not necessarily distinct) such that $ab = c$. For example, the empty set and the set $\{16, 20\}$ are product-free, whereas the sets $\{4, 16\}$ and $\{2, 8, 16\}$ are not product-free. Find the number of product-free subsets of the set $\{1, 2, 3, 4, ..., 7, 8, 9, 10\}$.

Challenge Problems

Challenge 1. Eleven (distinct) committees of size five are chosen from a group of people. Suppose that every pair of committees has exactly one common member. Prove that some person belongs to at least four committees.

Challenge 2. Let $S = \{1, 2, 3, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

Challenge 3. Let $|U|$, $\sigma(U)$, and $\pi(U)$ denote the number of elements, the sum, and the product, respectively, of a finite set U of positive integers. Let S be a finite set of positive integers. Prove that

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S)$$

for all integers $m \geq \sigma(S)$.