Prime Factorization

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The *prime factorization* of a positive integer is its expression as a product of primes. Prime factorizations can be used to compute the number of divisors of a positive integer, as well as the sum of its divisors.

When factoring large integers, identities such as a difference or sum of n^{th} powers and the Sophie Germain identity are helpful. When solving problems involving the divisors of an integer n, it is helpful to visualize the divisors as entries of an k-dimensional grid, where k is the number of distinct prime divisors of n.

Number of divisors formula. Prove that the number of divisors of a positive integer $n = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdot \ldots$ is

$$(e_1 + 1)(e_2 + 1)(e_3 + 1) \ldots.$$

Sum of divisors formula. Prove that the sum of the divisors of a positive integer $n = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdot \ldots$ is

$$\left(1 + 2 + 2^2 + \ldots + 2^{e_1}\right)\left(1 + 3 + 3^2 + \ldots + 3^{e_2}\right)\left(1 + 5 + 5^2 + \ldots + 5^{e_3}\right) \ldots.$$

Difference of powers. Let x and y be real numbers and n be a positive integer. Then

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \ldots + xy^{n-1} + y^n\right)$$

Sum of powers. Let x and y be real numbers and n be an odd positive integer. Then

$$x^n + y^n = (x + y)\left(x^{n-1} - x^{n-2}y + \ldots - xy^{n-1} + y^n\right)$$

Sophie Germain identity. Let x and y be real numbers. Then

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2).$$

Example 1. How many positive two-digit integers are factors of $2^{24} - 1$?

Example 2. Find the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}.

Example 3. Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

Example 4. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $\frac{n}{75}$.

Example 5. Let $d(n)$ denote the number of divisors of n, and let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

Find the unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$.

1 Problems

Problem 1. Find the prime factorization of 159999.

Problem 2. The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

Problem 3. For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

Problem 4. Find the number of ordered pairs of positive integers (m, n) such that $m^2n = 20^{20}$.

Problem 5. What is the smallest positive integer with six positive odd integer divisors and twelve positive even integer divisors?

Problem 6. What is the largest 2-digit prime factor of $\binom{200}{100}$?

Problem 7. Find the number of sets a, b, c of three distinct positive integers with the property that the product of $a, b,$ and c is equal to the product of $11, 21, 31, 41, 51, 61$.

Problem 8. Find the sum of the squares of the divisors of 210.

Problem 9. Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random with replacement from the set S and labeled $a_1, a_2,$ and a_3 in the order they are chosen. Find the probability that both a_1 divides a_2 and a_2 divides a_3.

Problem 10. Define the function f_1 on the positive integers by setting $f_1(1) = 1$ and if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of $n > 1$, then

$$f_1(n) = (p_1 + 1)^{e_1 - 1}(p_2 + 1)^{e_2 - 1} \cdots (p_k + 1)^{e_k - 1}.$$

For every $m \geq 2$, let $f_m(n) = f_1(f_{m-1}(n))$. For how many Ns in the range $1 \leq N \leq 400$ is the sequence $(f_1(N), f_2(N), f_3(N), \dots)$ unbounded?

Problem 11. For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2 - 1)!}{(n!)^n}$$

an integer?

Problem 12. Find the largest prime factor of $13^4 + 16^5 - 172^2$, given that it is the product of three distinct primes.

Challenge Problems

Challenge 1. Prove that

$$\frac{\gcd(a,b,c)^2}{\gcd(a,b)\gcd(a,c)\gcd(b,c)} = \frac{\mathrm{lcm}(a,b,c)^2}{\mathrm{lcm}(a,b)\mathrm{lcm}(a,c)\mathrm{lcm}(b,c)}.$$

Challenge 2. Prove that

$$(n+1)\cdot \mathrm{lcm}\left(\binom{n}{0},\binom{n}{1},\dots,\binom{n}{n}\right) = \mathrm{lcm}(1,2,\dots,n+1).$$

Challenge 3. Let s_n be the smallest positive integer with exactly n divisors. Prove that s_{2^k} is a factor of $s_{2^{k+1}}$ for all positive integers k.