Logarithms

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Logarithms are the inverse of exponentiation; that is, $\log_b x$ is defined to be the number such that, when b is raised to the power of it, equals x.

Properties of logarithms. For a positive real number $b \neq 1$ (known as the *base*) and positive real numbers x and y,

$$\log_b b^n = n$$

$$\log_b x + \log_b y = \log_b xy$$

$$\log_x y = \frac{\log_b y}{\log_b x}.$$

Example 1. Find x if $\log_{3x} 4 = \log_{2x} 8$.

Example 2. Which of the following is the value of $\sqrt{\log_2 6 + \log_3 6}$?

(A) 1

(B) $\sqrt{\log_5 6}$

(C) 2

(D) $\sqrt{\log_2 3} + \sqrt{\log_3 2}$

(E) $\sqrt{\log_2 6} + \sqrt{\log_3 6}$

Example 3. What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

Example 4. Suppose a real number $x > 1$ satisfies

$$\log_2(\log_4 x) + \log_4(\log_{16} x) + \log_{16}(\log_2 x) = 0.$$

Compute

$$\log_2(\log_{16} x) + \log_{16}(\log_4 x) + \log_4(\log_2 x).$$

Problems

Problem 1. What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdot \ldots \cdot \log_{21} 25 \cdot \log_{23} 27?$$

Problem 2. For how many integral values of x can a triangle of positive area be formed having side lengths $\log_2 x, \log_4 x, 3$?

Problem 3. What is the value of

$$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}?$$

Problem 4. Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $(\log_2 \frac{x}{y})^2$?

Problem 5. The sequence

$$\log_{12} 162, \log_{12} x, \log_{12} y, \log_{12} z, \log_{12} 1250$$

is an arithmetic progression. What is x?

Problem 6. What is the value of a for which $\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1$?

Problem 7. Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b} = 100$$

and all four terms on the left are positive integers, where log denotes the base-10 logarithm. What is ab?

Problem 8. Positive integers a and b satisfy the condition

$$\log_2 \left(\log_{2^a} \left(\log_{2^b} \left(2^{1000}\right)\right)\right) = 0.$$

Find the sum of all possible values of $a + b$.

Problem 9. There is a unique positive real number x such that the three numbers $\log_8(2x), \log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. Find x.

Problem 10. Let x, y, and z be real numbers satisfying the system

$$\begin{aligned}\log_2(xyz - 3 + \log_5 x) &= 5,\\ \log_3(xyz - 3 + \log_5 y) &= 4,\\ \log_4(xyz - 3 + \log_5 z) &= 4.\end{aligned}$$

Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.

Problem 11. Define binary operations $\diamondsuit$ and $\heartsuit$ by

$$a \diamondsuit b = a^{\log_7(b)} \qquad \text{and} \qquad a \heartsuit b = a^{\frac{1}{\log_7(b)}}$$

for all real numbers a and b for which these expressions are defined. The sequence (a_n) is defined recursively by $a_3 = 3 \heartsuit 2$ and

$$a_n = (n \heartsuit (n-1)) \diamondsuit a_{n-1}$$

for all integers $n \geq 4$. To the nearest integer, what is $\log_7(a_{2019})$?

Problem 12. In a Martian civilization, all logarithms whose bases are not specified as assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x) = 56$$

$$\log_{\log x}(x) = 54$$

and finds that this system of equations has a single real number solution $x > 1$. Find b.

Problem 13. Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

Problem 14. Let x, y, and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

Find xy^5z.

Challenge Problems

Challenge 1. Let $a > 1$ and $x > 1$ satisfy

$$\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$$

and

$$\log_a(\log_a x) = 256.$$

Find x.

Challenge 2. Every year, CMU's Decision Science majors take a famous multiple choice test. Each question has five choices, but instead of bubbling in one choice, you write a *probability* beside each choice. For each question, your 5 probabilities must sum to 1. For each question, you score $\log(x)$ points, where x is the value you wrote beside the correct choice.

What is the optimal strategy for this test? Assume that you're not sure about some questions and you only want to maximize the expected value of your score.

Challenge 3. Let $n \in \mathbb{Z}^+$. There are $n+1$ boxes in a row, and the leftmost box contains n stones. At every move, a stone in a box with k stones moved right by at most k squares. Prove that the minimum number of moves needed to move all n stones to the rightmost box is $\Theta(n \log n)$.