Large Numbers

Holden Mui

Name: ______________________________

Date: ______________________________

A collection of problems that have large numbers in them.

Problem 1. Six boxes in a row each initially contain one coin. If possible, you repeatedly either

- remove a coin from some box and add two coins to the box to the right of it, or
- remove a coin from some box and swap the contents of the two boxes immediately to the right of it.

Explain why the process must terminate. Additionally, try to make as many coins as you can appear in the rightmost box.

Problem 2. Let $\oplus$ denote concatenation. A sequence of positive integers $s_2, s_3, \ldots$ is defined as:

- $s_{i+1} = \frac{1}{10}s_i$ if s_i ends in zero, else
- $s_{i+1} = L \oplus (R-1) \oplus (R-1)$, where $L \oplus R = n$, L's units digit is less than R's units digit, and R's digits are all at least n's units digit.

For example, if $s_2 = 17151345543$, then $s_3 = 17151345542345542$. Prove that the sequence must terminate.

Problem 3. Define the *hereditary base b representation* of a positive integer n recursively to be the representation of n in base b, where all the exponents in the representation are also written in hereditary base b. For example,

$$35 = 2^{2^{2^1}+1} + 2^1 + 1$$

in hereditary base-2 notation, and

$$100 = 3^{3^1+1} + 2 \cdot 3^2 + 1$$

in hereditary base-3 notation.

A *Goodstein sequence* is a sequence of positive integers $s_2, s_3, \ldots$ for which the hereditary base $i+1$ representation of $s_{i+1} + 1$ is the hereditary base i representation of s_i, but with all i's replaced with $i+1$'s. Prove that every Goodstein sequence terminates.

Problem 4. A sequence of positive integers $s_2, s_3, \ldots$ is defined as:

- $s_{i+1} = \frac{1}{10}s_i$ if s_i ends in zero,
- else
$$s_{i+1} = L \oplus \underbrace{(R-1) \oplus \ldots \oplus (R-1)}_{i \text{ times}},$$
where $L \oplus R = n$, L's units digit is less than R's units digit, and R's digits are all at least n's units digit.

Prove that the sequence must terminate.