Combinatorial Games

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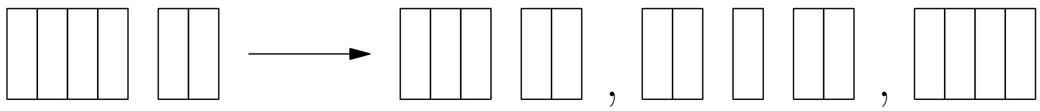
A *deterministic game* is a two-player game that

- is deterministic; that is, has no element of randomness, and
- has perfect information; that is, no information is ever kept a secret for any player.

The goal of problems involving combinatorial games is often to determine the winning player. Symmetry often plays an integral role in the solution of such problems – oftentimes by mirroring an opponent's move. If symmetry cannot be used, it is helpful to make a table of the player with the winning strategy for each possible configuration.

Example 1. There is a pile of n stones, and Alice and Bob alternate taking either 1, 2, or 3 stones from the pile at a time. A player loses if they cannot make a move. For which n does Bob have a winning strategy?

Example 2. Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3, 2), (2, 1, 2), (4), (4, 1), (2, 2)$, or $(1, 1, 2)$.



Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

(A) $(6, 1, 1)$

(B) $(6, 2, 1)$

(C) $(6, 2, 2)$

(D) $(6, 3, 1)$

(E) $(6, 3, 2)$

Example 3. On an m by n board, a token is placed in one of the corners. Luke and Edward alternate moving the token to an adjacent square, with Luke going. It's forbidden for the token to visit a square it has visited before, and the player who cannot move loses. Find all pairs (m, n) for which Luke has a winning strategy.

Example 4. Elmo and Elmo's clone are playing a game. Initially, n points are given on a circle. On a player's turn, that player must draw a triangle using three unused points as vertices, without creating any crossing edges. The first player who cannot move loses. If Elmo goes first and players alternate turns, find the sum of all integers $3 \leq n \leq 16$ for which Elmo has a winning strategy.

Example 5. There is a pile of n stones, and Alice and Bob alternate taking either 1, 2, or 4 stones from the pile at a time. A player loses if they cannot make a move. Find the sum of all $n \leq 20$ for which Bob has a winning strategy.

Problems

Problem 1. There is a pile of 100 stones, and Alice and Bob alternate taking either 1, 3, or 4 stones from the pile at a time. A player loses if they cannot make a move. If Alice goes first, who has a winning strategy?

Problem 2. Serena and Christine have an $m \times n$ chocolate bar, where m and n are positive integers less than 10. Serena moves first, and the players take turns breaking one of the pieces of the chocolate into two smaller pieces along the grid of the chocolate bar; the last player to make a move wins. For how many pairs (m, n) does Serena have a winning strategy?

Problem 3. Dan and Sam play a game on an 8×8 grid. They alternate placing down nonoverlapping dominoes on the grid such that each domino covers exactly two adjacent squares, with Dan going first. A player loses if they cannot make a move. Who has the winning strategy?

Problem 4. Bela and Jenn play the following game on the closed interval $[0, n]$ of the real number line, where n is a fixed integer greater than 4. They take turns playing, with Bela going first. At his first turn, Bela chooses any real number in the interval $[0, n]$. Thereafter, the player whose turn it is chooses a real number that is more than one unit away from all numbers previously chosen by either player. A player unable to choose such a number loses. Using optimal strategy, which player will win the game?

(A) Bela will always win.

(B) Jenn will always win.

(C) Bela will win if and only if n is odd.

(D) Jenn will win if and only if n is odd.

(E) Jenn will win if and only if $n > 8$.

Problem 5. Barbara and Jenna play the following game, in which they take turns. A number of coins lie on a table. When it is Barbara's turn, she must remove 2 or 4 coins, unless only one coin remains, in which case she loses her turn. When it is Jenna's turn, she must remove 1 or 3 coins. A coin flip determines who goes first. Whoever removes the last coin wins the game. Assume both players use their best strategy. Who will win when the game starts with 2013 coins and when the game starts with 2014 coins?

Problem 6. There are two piles of stones, with 20 stones and 21 stones, respectively. A move consists of taking a nonzero number of stones from one pile or taking the same number of stones from both piles. William and Gopal alternate moves, with William going first. What first move guarantees William a winning strategy?

Problem 7. Vincent and Brandon play a game on a convex polygon of n sides. They alternate drawing a diagonal of the polygon that does not intersect any previously drawn diagonals, with Vincent going first. A player loses if they cannot make a move. Characterize all n for which Vincent has a winning strategy.

Problem 8. Ashley and Elizabeth play a game. There are 9 cards numbered 1 through 9 on a table, and the players alternate taking the cards, with Ashley going first. A player wins if at any point they hold three cards with sum 15; if all nine cards are taken before this occurs, the game is a tie. Does either player have a winning strategy?

Challenge Problems

Challenge 1. A row of fifty coins with integer denominations is given, such that the sum of the denominations is odd. Colin and Daniel alternate taking either coin at the left end of the row or the right end of the row, with Colin playing first. The player with the most money at the end wins. Which player has the winning strategy?

Challenge 2. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

Challenge 3. Mr. Kim and Mrs. Moore play a game on the unit edges of an infinite square lattice, making moves in turn. Mr. Kim makes the first move. A move consists of orienting any unit edge of the grid that has not been oriented before. If at some stage, some oriented edges form an oriented cycle, Mrs. Moore wins. Does Mrs. Moore have a strategy that guarantees her a win?