Divisibility

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In number theory, problems involving divisibility constraints can be usually be solved with prime factorizations, divisibility rules, modular arithmetic, or (more rarely) size bounding.

Example 1. Prove the divisibility rules for the number 2, 3, . . ., 10.

Example 2. Prove that the number of divisors of a positive integer $n = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdot \ldots$ is

$$(e_1 + 1)(e_2 + 1)(e_3 + 1) \ldots .$$

Example 3. Find the number of positive integers that are divisors of at least one of $10^{10}, 15^7, 18^{11}$.

Example 4. What is that largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

1 Problems

Problem 1. The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $PQRST$. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?

Problem 2. Find the sum of all positive two-digit integers that are divisible by each of their digits.

Problem 3. Find the number of positive integers with three not necessarily distinct digits, abc, with $a \neq 0$ and $c \neq 0$ such that both abc and cba are multiples of 4.

Problem 4. The base-ten representation for 19! is

$$121, 6T5, 100, 40M, 832, H00,$$

where T, M, and H denote digits that are not given. What is $T + M + H$?

Problem 5. Find the number of ordered pairs of positive integers (m, n) such that $m^2 n = 20^{20}$.

Problem 6. What is the largest 2-digit prime factor of $\binom{200}{100}$?

Problem 7. Find the number of five-digit positive integers, n, that satisfy the following conditions:

- the number n is divisible by 5,
- the first and last digits of n are equal, and
- the sum of the digits of n is divisible by 5.

Problem 8. Find the smallest positive integer k such that $1^2+2^2+3^2+\ldots+k^2$ is a multiple of 200.

Problem 9. Find the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}.

Problem 10. Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N. Find the sum of all the digits of all the numbers in S.

Problem 11. Let S be the set of integers between 1 and 2^{40} whose binary expansions have exactly two 1's. If a number is chosen at random from S, find the probability that it is divisible by 9.

Problem 12. Find the largest prime factor of $13^4 + 16^5 - 172^2$, given that it is the product of three distinct primes.

Problem 13. Find the sum of all positive integers n for which $2n - 3$ divides $15 \cdot n!^2$.

2 Challenge Problems

Challenge 1. Let s_n be the smallest positive integer with exactly n divisors. Prove that s_{2^k} is a factor of $s_{2^{k+1}}$ for all positive integers k.

Challenge 2. Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

Challenge 3. Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:

- Each number in the table is congruent to 1 modulo n.
- The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2.

Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the number in the j^{th} column. Prove that $(R_1 + \ldots R_n) - (C_1 + \ldots C_n)$ is divisible by n^4.