Diophantine Equations

Holden Mui

Name: ______________________________

Date: ______________________________

A *Diophantine equation* is an algebraic equation in which the solutions of interest are those for which all variables are integers. There are two main techniques for solving Diophantine equations.

- Factoring a Diophantine equation can reduce it to a finite case check.
- Bounding a Diophantine equation restricts the size of a variable, thus reducing the equation to several small cases.

Example 1. There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p.

Example 2. Find all positive integers x such that there exists a positive integer y satisfying

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{7}.$$

Example 3. Find the sum of all positive integers n for which $n^2 - 19n + 99$ is a perfect square.

Example 4. Find all triples of positive integers (x, y, z) for which

$$x^3 + y^3 + z^3 - 3xyz = 11.$$

Problems

Problem 1. Find all pairs of positive integers (x, y) such that

$$x^2 - y^2 = 23.$$

Problem 2. Let A, M, and C be digits with

$$(100A + 10M + C)(A + M + C) = 2005$$

What is A?

Problem 3. Find the prime number p such that $71p + 1$ is a perfect square.

Problem 4. There exist unique positive integers x and y that satisfy the equation $x^2 + 84x + 2008 = y^2$. Find $x + y$.

Problem 5. Find all triples (a, b, c) of positive integers which satisfy the simultaneous equations

$$\begin{aligned} ab + bc &= 44 \\ ac + bc &= 23 \end{aligned}$$

Problem 6. Find all pairs of integers (x, y) for which $4^y - 615 = x^2$.

Problem 7. Find all pairs of positive integers (a, b) for which $a^2 + b$ exceeds $a + b^2$ by 36.

Problem 8. Find n such that $n - 76$ and $n + 76$ are both cubes of positive integers.

Problem 9. Find all pairs of positive integers (m, n) for which

$$n^4 - 7n^2 + 1 = m^2.$$

Problem 10. Find $3x^2y^2$ if x and y are integers such that

$$y^2 + 3x^2y^2 = 30x^2 + 517.$$

Problem 11. If p, q, and r are primes with $pqr = 7(p+q+r)$, find $p+q+r$.

Problem 12. Find all right triangles with integer side lengths whose perimeter and area are equal.

Problem 13. Find all triples of positive integers (p, q, r) for which a regular p-gon, a regular q-gon, and a regular r-gon can meet at a corner such that they don't overlap and there are no gaps.

Problem 14. Find all pairs of integers (x, y) that satisfy the equation

$$2(x^2 + y^2) + x + y = 5xy.$$

Challenge Problems

Challenge 1. Show that there is no triple of positive integers (a, b, c) for which $a^2 + b + c$, $b^2 + c + a$, and $c^2 + a + b$ are all square.

Challenge 2. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

Challenge 3. Find all sets $\{a_1, a_2, a_3, a_4\}$ of four distinct positive integers such that there are four pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides $a_1 + a_2 + a_3 + a_4$.

Challenge 4. Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

Challenge 5. Find all triples (a, b, c) of positive integers such that

$$a^3 + b^3 + c^3 = (abc)^2.$$