Three-Dimensional Geometry

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As opposed to two-dimensional geometry, three-dimensional geometry is the geometry of space. Three-dimensional geometry problems can often by solved by taking appropriate cross sections of the configuration and using two-dimensional techniques on the resulting problem to solve the problem. The Pythagorean theorem is also a useful tool when dealing with configurations with lengths and right-angled intersections.

Volume formulae.

- Sphere with radius r: $\frac{4}{3}\pi r^3$
- Prism with height h and base area B: Bh
- Pyramid with height h and base area B: $\frac{1}{3}Bh$
- Frustum with height h and base areas B_1 and B_2: $\frac{B_1+\sqrt{B_1B_2}+B_2}{3} \cdot h$
- Regular tetrahedron with side s: $\frac{\sqrt{2}}{12}s^3$

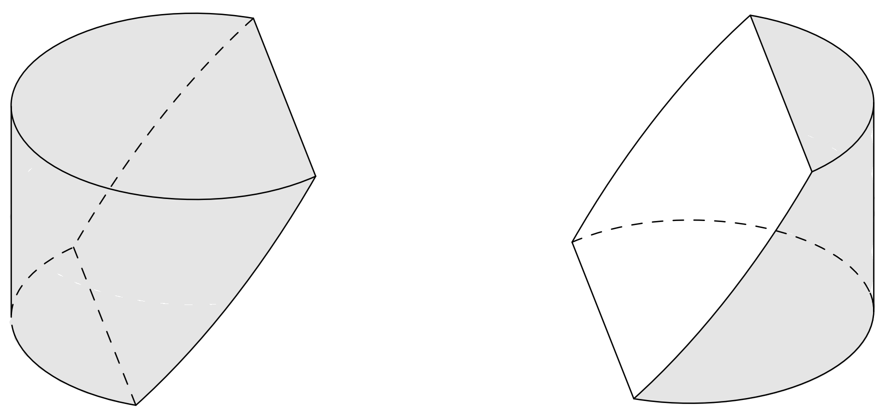
Surface area formulae.

- Sphere with radius r: $4\pi r^2$
- Cone with radius r and slant height ℓ: $\pi r^2 + \pi \ell h$

Example 1. Two congruent right circular cones each with base radius 3 and height 8 have the axes of symmetry that intersect at right angles at a point in the interior of the cones a distance 3 from the base of each cone. What is the largest possible radius of a sphere that can lie within both cones?

Example 2. Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

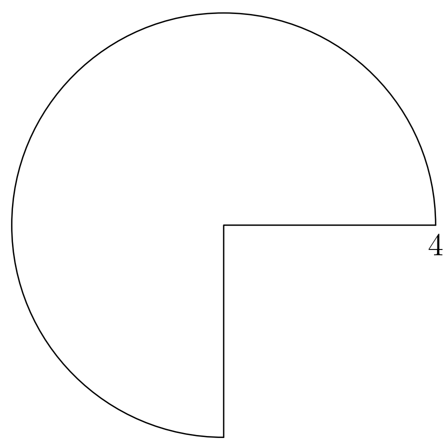
Example 3. A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge of one of the circular faces of the cylinder so that arc AB on that face measures $120°$. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. Find the area of one of these unpainted faces.



Example 4. Consider the paper triangle whose vertices are $(0,0)$, $(34,0)$, and $(16,24)$. The vertices of its midpoint triangle are the midpoints of its sides. A triangular pyramid is formed by folding the triangle along the sides of its midpoint triangle. What is the volume of this pyramid?

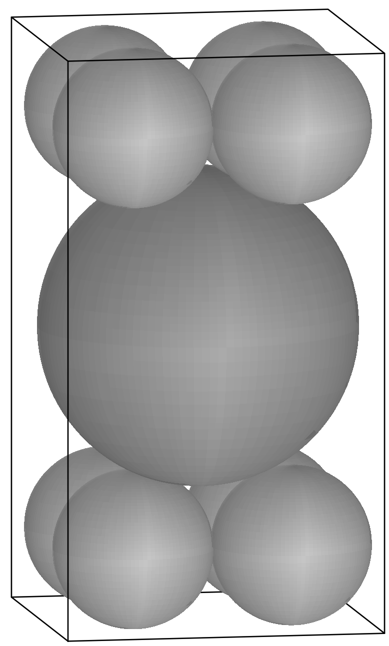
Problems

Problem 1. A three-quarter sector of a circle of radius 4 inches together with its interior can be rolled up to form the lateral surface of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic inches?



Problem 2. Consider the set of points that are inside or within one unit of a rectangular parallelepiped (box) that measures 3 by 4 by 5 units. Find the volume of this set.

Problem 3. A $4 \times 4 \times h$ rectangular box contains a sphere of radius 2 and eight smaller spheres of radius 1. The smaller spheres are each tangent to three sides of the box, and the larger sphere is tangent to each of the smaller spheres. What is h?



Problem 4. A sphere with center O has radius 6. A triangle with sides of length $15, 15$, and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

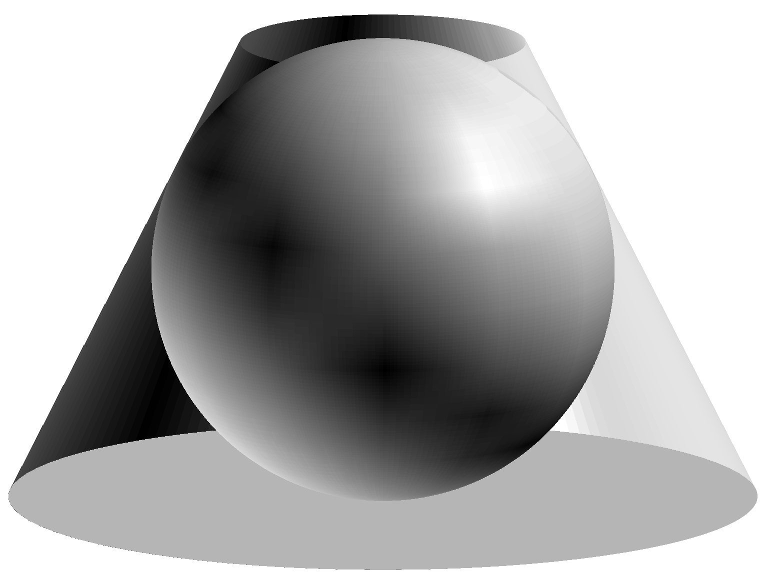
Problem 5. Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

Problem 6. A pyramid has a triangular base with side lengths 20, 20, and 24. The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length 25. Find the volume of the pyramid.

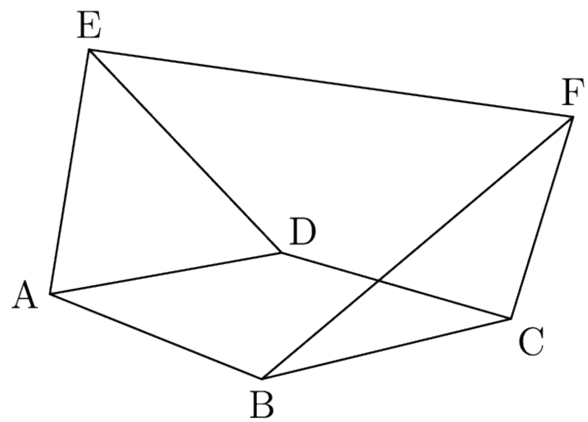
Problem 7. A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. Find the radius of the spheres.

Problem 8. A cylindrical barrel with radius 4 feet and height 10 feet is full of water. A solid cube with side length 8 feet is set into the barrel so that the diagonal of the cube is vertical. Find the volume of displaced water.

Problem 9. A sphere is inscribed in a truncated right circular cone as shown. The volume of the truncated cone is twice that of the sphere. What is the ratio of the radius of the bottom base of the truncated cone to the radius of the top base of the truncated cone?



Problem 10. The solid shown has a square base of side length s. The upper edge is parallel to the base and has length $2s$. All other edges have length s. Given that $s = 6\sqrt{2}$, what is the volume of the solid?



Problem 11. Tetrahedron $ABCD$ has $AD = BC = 28$, $AC = BD = 44$, and $AB = CD = 52$. For any point X in space, define $f(X) = AX + BX + CX + DX$. Find the least possible value of $f(X)$.

Problem 12. In a rectangular box $ABCDEFGH$ with edge lengths $AB = AD = 6$ and $AE = 49$, a plane slices through point A and intersects edges $\overline{BF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HD}$ at points P, Q, R, S respectively. Given that $AP = AS$ and $PQ = QR = RS$, find the area of pentagon $APQRS$.

Challenge Problems

Challenge 1. What is the maximum number of edges of a regular octahedron can one see at a time?

Challenge 2. A box fits inside another box. Must the sum of the edge lengths of the smaller box be less than the sum of the edge lengths of the larger box?

Challenge 3. Points A, B, C, and D lie in space such that a sphere is tangent to $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Prove that these four tangency points are coplanar.

Challenge 4. Let $P_1P_2P_3P_4$ be a tetrahedron in $\mathbb{R}_3$ and let O be its circumcenter. There exists a point H such that for each i, $\overline{P_iH}$ is perpendicular to the plane through the other three vertices. $\overline{P_1H}$ intersects the plane through P_2, P_3, P_4 at A, and contains a point $B \neq P_1$ such that $OP_1 = OB$. Show that $HB = 3HA$.