Introduction to Proof

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A *proof* is a convincing argument that demonstrates why a mathematical statement is true. There are several standard methods of proof, such as proof by example, proof by exhaustion, proof by contradiction, and proof by induction.

Proofs are usually written using complete sentences in paragraph format. One way to write a proof is to first think about what you would say to someone who didn't know how to solve the problem, and then to write down these sentences.

Problem 1.

(a) Prove or disprove: there exist positive integers a and b for which $a^2 + b^2 = 108$.

(b) Prove or disprove: there exist positive integers a and b for which $a^2 + b^2 = 109$.

Problem 2. A number is *even* if it equals $2k$ for some integer k. A number is *odd* if it equals $2k + 1$ for some integer k. Using these definitions, prove that:

(a) the sum of two even numbers is even

(b) the sum of an even number and an odd number is odd

(c) the sum of two odd numbers is even.[1]

Problem 3.

(a) Prove that the square of any odd number is one more than a multiple of 4.

(b) Is there a square number that is one less than a multiple of 4?

[1]In general, statements like these are considered obvious and don't require justification. However, the purpose of this problem is to have practice with definitions.

Problem 4. A *rational number* is a number that can be expressed as a fraction whose numerator and denominator are integers.

(a) Is 108 a rational number?

(b) Is 0.4 a rational number?

(c) Recall that π is the ratio of a circle's circumference c to its diameter d; that is, $\pi = \frac{c}{d}$. Does this mean π is a rational number?

(d) Prove that the sum of two rational numbers is rational.

A direct proof of a mathematical result is a proof that starts from true statements and uses logical deductions to arrive at the mathematical result. All the statements we just proved were proved via a direct proof.

There is another method of proof, called *proof by contradiction.* It starts by assuming the hypothesis is false and using this assumption to derive a contradiction. In this way, the hypothesis is proved to be true.

Problem 5. Prove that if a and b are positive integers with $a + b \geq 9$, then at least one of $a \geq 5$ and $b \geq 5$ is true.

Problem 6. Prove that if n is a positive integer for which n^2 is even, then n must be even.

Problem 7. Prove that the sum of a rational number and an irrational number is irrational.

The following problems are famous mathematical results.

Problem 8. Prove that $\sqrt{2}$ is irrational.

Problem 9 (Pythagorean Theorem). Let ABC be a right triangle with a right angle at C. Prove that

$$AB^2 = AC^2 + BC^2.$$