Modular Arithmetic

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Modular arithmetic is arithmetic that deals with remainders after division by a fixed number. The remainders can be thought of as "wrapping around" once a certain value, known as the *modulus*, is exceeded. A familiar problem of modular arithmetic is *clock arithmetic*, where statements such as 19:00 + 8 hours = 3:00 make sense.

Problem 1. Compute the remainder when the following numbers are divided by 12:

(a) 31

(b) 108

(c) 12345

(d) 54321

Problem 2. It is currently 5pm in the evening. What time will it be:

(a) 10 hours from now?

(b) 100 hours from now?

(c) 1000 hours from now?

(d) 1000000 hours from now?

Problem 3. During class, Mark writes his last name (Saengrungkongka) over and over again on a sheet of paper. He completes 777 letters before the paper is taken away by his teacher and he is reminded to pay attention in class.

(a) What is the last letter he writes?

(b) What letter is in the middle position?

Problem 4. In the decimal expansion of $\frac{3}{7}$, what is:

(a) the hundredth digit to the right of the decimal point?

(b) the thousandth digit to the right of the decimal point?

Problem 5. In the decimal expansion of $\frac{1}{37}$, what is:

(a) the hundredth digit to the right of the decimal point?

(b) the thousandth digit to the right of the decimal point?

Problem 6. Find the units digits of the following numbers:

(a) 3^{14}

(b) 3^{999}

(c) 7^{14}

(d) 7^{999}

(e) 13^{14}

(f) 17^{14}

Problem 7. Find the last two digits of the following numbers:

(a) 7^{14}

(b) 7^{999}

Problem 8. Compute the following expressions modulo 12:

(a) $108 + 109 + 110 + 111$

(b) $11108 + 11109 + 11110 + 11111$

(c) $13 \times 14 \times 15$

(d) $113 \times 114 \times 115$

(e) $1234 + 567 - 89$

(f) $19 \times 199 \times 1999$

Problem 9. Sangay has 44 boxes of soda in his truck. The cans of soda in each box are packed oddly so that there are 113 cans of soda in each box. Sangay plans to pack the sodas into cases of 12 cans to sell. After making as many complete cases as possible, how many sodas will Sangay have leftover?

> We say that $a \equiv b \pmod{n}$ if and only if $a - b$ is a multiple of n. The number n is called the *modulus*.

Problem 10. Let a and b be positive integers such that $a \equiv b \pmod{n}$, and and let c and d be positive integers such that $c \equiv d \pmod{n}$.

(a) Prove that $a + c \equiv b + d \pmod{n}$.

(b) Prove that $ac \equiv bd \pmod{n}$.

(c) Is it true that $a^c \equiv b^d \pmod{n}$?

Problem 11.

(a) What is the remainder when $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$ is divided by 9?

(b) What is the remainder when $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$ is divided by 10?

Problem 12.

(a) Prove that the square of any odd number is one more than a multiple of 8.

(b) Prove that the square of any number is at most one away from a multiple of 5.

(c) Prove that the cube of any number is at most one away from a multiple of 7.

(d) Prove that the cube of any number is at most one away from a multiple of 9.

Problem 13. Find the last three digits of

$$9 \times 99 \times 999 \times 9999 \times \ldots \times \underbrace{99\ldots9}_{999\text{ 9's}}.$$

Problem 14. Prove the sum of the digits of every multiple of 9 is also a multiple of 9.

Problem 15.

(a) Prove that the product of any four consecutive integers is divisible by 8.

(b) Prove that the product of any four consecutive integers is divisible by 3.[1]

Problem 16. Can the sum of two squares be congruent to 3 modulo 4?

[1](a) and (b) combined implies that the product of any four consecutive integers is divisible by 24.

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