Midterm 2 Solutions

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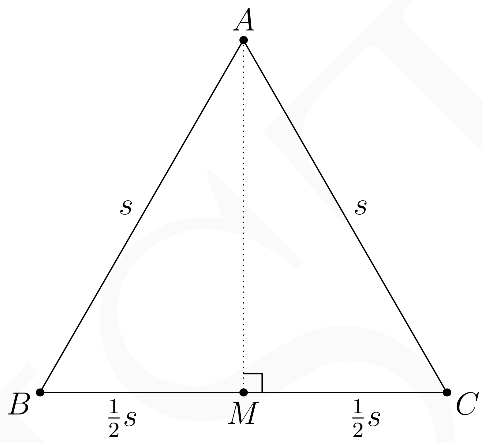
Problem 1. Determine, with proof, the greatest common divisor of 1971 and 10001.

Solution. By the Euclidean algorithm,

$$\begin{aligned}
\gcd(1971, 10001) &= \gcd(1971, 10001 - 5 \cdot 1971) \\
&= \gcd(1971, 146) \\
&= \gcd(1971 - 13 \cdot 146, 146) \\
&= \gcd(73, 146) \\
&= \boxed{73}.
\end{aligned}$$

Problem 2. Prove that the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$.

Solution A. Let ABC be an equilateral triangle with side length s, and let M be the midpoint of $\overline{BC}$.



Since $AB = AC$, $\overline{AM}$ is perpendicular to $\overline{BC}$. By the Pythagorean theorem,

$$AM = \sqrt{AC^2 - CM^2} = \sqrt{s^2 - (\tfrac{1}{2}s)^2} = \sqrt{s^2 - \tfrac{1}{4}s^2} = \sqrt{\tfrac{3}{4}s^2} = \tfrac{\sqrt{3}}{2}s.$$

By the triangle area formula, the area of triangle ABC is

$$\tfrac{1}{2} \cdot BC \cdot AM = \tfrac{1}{2} \cdot s \cdot \tfrac{\sqrt{3}}{2}s = \tfrac{\sqrt{3}}{4}s^2.$$

Solution B. The semiperimeter is $s = \frac{1}{2}(s+s+s) = \frac{3}{2}s$. Thus by Heron's formula, the area is

$$\sqrt{(\tfrac{3}{2}s)(\tfrac{3}{2}s - s)(\tfrac{3}{2}s - s)(\tfrac{3}{2}s - s)} = \sqrt{(\tfrac{3}{2}s)(\tfrac{1}{2}s)(\tfrac{1}{2}s)(\tfrac{1}{2}s)} = \sqrt{\tfrac{3}{16}s^4} = \tfrac{\sqrt{3}}{4}s^2.$$

Problem 3. Find, with proof, all pairs of real numbers (x, y) for which $x + \frac{1}{y} = 2$ and $y + \frac{1}{x} = \frac{9}{4}$.

Solution A. The first equation rearranges to $x = 2 - \frac{1}{y}$. Substituting into the second equation gives $y + \frac{1}{2-1/y} = \frac{9}{4}$, which is equivalent to $y + \frac{y}{2y-1} = \frac{9}{4}$. Multiplying both sides by $4(2y-1)$ gives

$$4(2y-1)y + 4y = 9(2y-1),$$

which simplifies to the quadratic $8y^2 - 18y + 9 = 0$. Since this factors as $(2y-3)(4y-3)$, the two solutions for y are $y = \frac{3}{2}$ and $y = \frac{3}{4}$.

- If $y = \frac{3}{2}$, then $x = 2 - \frac{1}{3/2} = \frac{4}{3}$. This gives $(x, y) = (\frac{4}{3}, \frac{3}{2})$.
- If $y = \frac{3}{4}$, then $x = 2 - \frac{1}{3/4} = \frac{2}{3}$. This gives $(x, y) = (\frac{2}{3}, \frac{3}{4})$.

Therefore the answers are $\boxed{(\frac{2}{3}, \frac{3}{4}) \text{ and } (\frac{4}{3}, \frac{3}{2})}$, which both satisfy the original equation.

Solution B. The equations are equivalent to $\frac{xy+1}{y} = 2$ and $\frac{xy+1}{x} = \frac{9}{4}$. Since both equations are nonzero, dividing them is valid and gives

$$\frac{x}{y} = \frac{2}{9/4} = \frac{8}{9}.$$

Thus, $x = 8k$ and $y = 9k$ for some real number k. Substituting these values into the first equation yields $8k + \frac{1}{9k} = 2$. Multiplying both sides by $9k$ gives

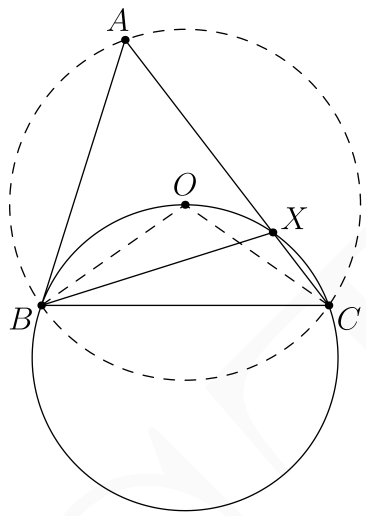
$$72k^2 + 1 = 18k,$$

which factors as $(12k-1)(6k-1) = 0$. Hence $k = \frac{1}{12}$ or $\frac{1}{6}$.

- If $k = \frac{1}{12}$, then $x = 8 \cdot \frac{1}{12} = \frac{2}{3}$ and $y = 9 \cdot \frac{1}{12} = \frac{3}{4}$.
- If $k = \frac{1}{6}$, then $x = 8 \cdot \frac{1}{6} = \frac{4}{3}$ and $y = 9 \cdot \frac{1}{6} = \frac{3}{2}$.

Therefore the answers are $\boxed{(\frac{2}{3}, \frac{3}{4}) \text{ and } (\frac{4}{3}, \frac{3}{2})}$, which both satisfy the original equation.

Problem 4. Let ABC be an acute triangle with circumcenter O such that $AC > BC$. The circumcircle of triangle BOC intersects $\overline{AC}$ again at X. Prove that $AX = BX$.



Solution. Let $\angle BAX = x$. By the inscribed angle theorem on triangle ABC, $\angle BOC = 2x$. By the inscribed angle theorem on quadrilateral $BOXC$, $\angle BXC = 2x$. Since $\angle AXC$ is a straight angle, $\angle AXB = 180^\circ - 2x$. Since the angles in triangle ABX sum to 180°,

$$\angle ABX = 180^\circ - (180^\circ - 2x) - x = x.$$

Therefore $\angle ABX = \angle BAX$, so $AX = BX$.

Problem 5. Prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

for every positive integer n.

Solution A. Induct on n. The base case $n=1$ is true because $\frac{1}{1\cdot 2}=\frac{1}{1+1}$. For the inductive step, assume the statement is true for $n=k$, so

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k\cdot(k+1)}=\frac{k}{k+1}.$$

The goal is to prove the statement for $n=k+1$. To do this, write

$$\begin{aligned}&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k\cdot(k+1)}+\frac{1}{(k+1)\cdot(k+2)}\\ &=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2},\end{aligned}$$

so the statement is true for $n=k+1$. This completes the induction.

Solution B. For any positive integer k,

$$\frac{1}{k(k+1)}=\frac{(k+1)-k}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}.$$

Therefore

$$\begin{aligned}&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}\\ &=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &=\frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\cdots+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}\\ &=\frac{1}{1}-\frac{1}{n+1}\\ &=\frac{n}{n+1}.\end{aligned}$$

Problem 6. Let r and s be distinct real numbers for which $r^2 = 1 + r$ and $s^2 = 1 + s$. Find, with proof, the value of $r^4 + s^4$.

Solution A. r and s are the roots of $x^2 - x - 1$. By Vieta's formulas, $r + s = 1$ and $rs = -1$. Therefore

$$r^2 + s^2 = (r+s)^2 - 2rs = 1^2 - 2(-1) = 3$$

and

$$r^4 + s^4 = (r^2 + s^2)^2 - 2(rs)^2 = 3^2 - 2(-1)^2 = \boxed{7}.$$

Solution B. Solving for r and s using the quadratic formula shows that r and s must equal $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Since these numbers square to $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$,

$$r^4 + s^4 = \left(\frac{3+\sqrt{5}}{2}\right)^2 + \left(\frac{3-\sqrt{5}}{2}\right)^2 = \frac{7+3\sqrt{5}}{2} + \frac{7-3\sqrt{5}}{2} = \boxed{7}.$$

Problem 7. Let F_n denote the n^{th} Fibonacci number. Prove that for all positive integers n, $\gcd(F_n, F_{n+1}) = 1$.

Solution. Induct on n. The base case $n = 1$ is true since

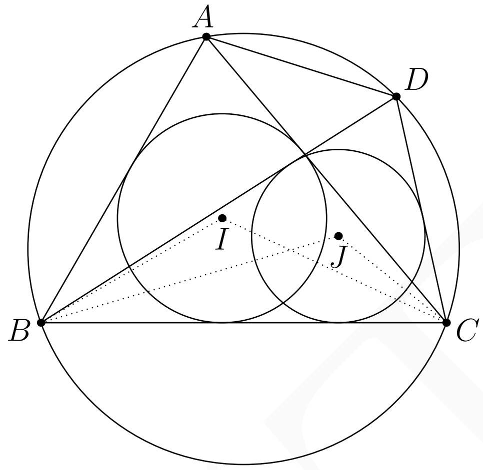
$$\gcd(F_1, F_2) = \gcd(1, 1) = 1.$$

For the inductive step, assume the statement is true for $n = k$, so that $\gcd(F_k, F_{k+1}) = 1$. The goal is to prove the statement for $n = k + 1$. By the Euclidean algorithm, the definition of the Fibonacci sequence, and the inductive hypothesis,

$$\gcd(F_{k+1}, F_{k+2}) = \gcd(F_{k+1}, F_{k+2} - F_{k+1}) = \gcd(F_{k+1}, F_k) = 1.$$

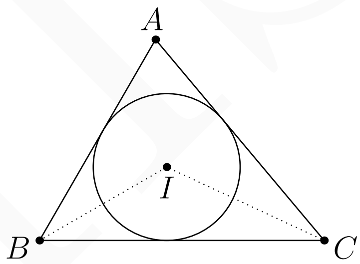
This completes the induction.

Problem 8. Let $ABCD$ be a cyclic quadrilateral, and let I and J be the incenters of triangles ABC and DBC. Prove that quadrilateral $BIJC$ is cyclic.



Solution. First, consider the following lemma.

Lemma. *Let triangle ABC have incenter I. Then $\angle BIC = 90^\circ + \frac{1}{2}\angle BAC$.*



Proof. Let $\angle ABC = b$ and $\angle ACB = c$. Then $\angle IBC = \frac{1}{2}b$ and $\angle ICB = \frac{1}{2}c$ since $\overline{BI}$ and $\overline{CI}$ are angle bisectors. Since the angles in triangle ABC sum to 180°, $\angle BAC = 180^\circ - b - c$, and since the angles in triangle IBC sum to 180°, $\angle BIC = 180^\circ - \frac{1}{2}b - \frac{1}{2}c$. Therefore

$$90^\circ + \tfrac{1}{2}\angle BAC = 90^\circ + \tfrac{1}{2}(180^\circ - b - c) = 180^\circ - \tfrac{1}{2}b - \tfrac{1}{2}c = \angle BIC. \quad \square$$

By the inscribed angle theorem, $\angle BAC = \angle BDC$. So two applications of the lemma gives

$$\angle BIC = 90^\circ + \tfrac{1}{2}\angle BAC = 90^\circ + \tfrac{1}{2}\angle BDC = \angle BJC.$$

Thus quadrilateral $BIJC$ must be cyclic by the inscribed angle theorem.

Problem 9. Determine the incorrect step in the following "proof" that $1 = 2$, and explain why that step is logically incorrect.

Step	Equation	Reasoning
Step 1	$a = b$	definition
Step 2	$a^2 = ab$	multiply by a
Step 3	$ab + a^2 = 2ab$	add ab
Step 4	$ab - a^2 = 2ab - 2a^2$	subtract $2a^2$
Step 5	$1(ab - a^2) = 2(ab - a^2)$	factor
Step 6	$1 = 2$	cancel $ab - a^2$

Solution. Step $\boxed{6}$ is logically incorrect because $ab - a^2$ equals zero and division by zero is not allowed.

Problem 10. Determine the incorrect step in the following "proof" that everyone in Bhutan has the same name, and explain why that step is logically incorrect.

The statement "any group of n people in Bhutan all have the same name" will be proved by induction on n.

(1) The base case $n = 1$ is true, because if there is only one person in the group, then all people in that group have the same name.

(2) For the inductive step, suppose the statement is true for $n = k$, so every group of k people in Bhutan all have the same name. Consider any group of $k + 1$ people; the goal is to prove that everyone in this group of $k + 1$ people has the same name.

(3) First, exclude one person (person A) and look at the remaining k people. By the inductive hypothesis, these k people all have the same name.

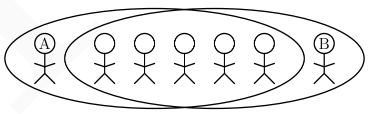
(4) Similarly, exclude some other person (person B) and look at the remaining k people. By the same reasoning, these k people all have the same name.

(5) Therefore, person A has the same name as all the $k - 1$ non-excluded people, who all have the same name as person B.

(6) Therefore person A, the $k - 1$ non-excluded people, and person B all have the same name.

(7) Thus everyone in the group of $k + 1$ people has the same name.

Hence, the statement "any group of n people in Bhutan all have the same name" is true. Applying this result to the group of all people in Bhutan shows that all people in Bhutan have the same name.



Solution. Step $\boxed{6}$ is logically incorrect because it is false when $k = 1$. When $k = 1$, there are no excluded people, so it is illogical to conclude that person A has the same name as person B.