Midterm 2 Review

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Name: ______________________________

Date: ______________________________

The second midterm exam will be held at 9:30am on Saturday, January 27, 2024. No calculators or notes are allowed during the exam. It is designed to measure both your progress learning the material and our ability to teach the material. Five of the problems will be problems you have already seen in class or on your homework, and five of the problems will be problems you have not seen before. You will have unlimited time for the exam.

The problems on this handout are intended to help you review for the exam. These problems will be similar in difficulty, style, and topic to the midterm exam problems.

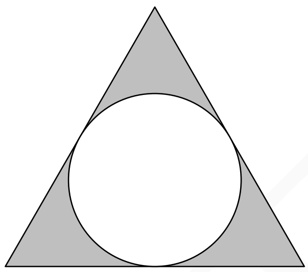
Problem 1. Let a, b, and c be real numbers satisfying

$$\begin{aligned} a + b + c &= 1 \\ a + \tfrac{1}{2}b + \tfrac{1}{4}c &= \tfrac{1}{8} \\ a + \tfrac{1}{3}b + \tfrac{1}{9}c &= \tfrac{1}{27}. \end{aligned}$$

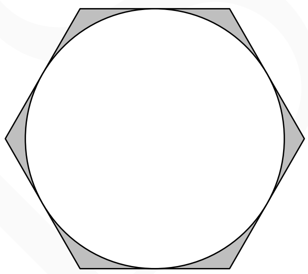
Find $a + \frac{1}{4}b + \frac{1}{16}c$.

Problem 2.

(a) The equilateral triangle below has a side length of 6. What is the area of the shaded region?



(b) The regular hexagon below has a side length of 6. What is the area of the shaded region?



Problem 3. Prove that

$$\frac{1}{2^2-1}+\frac{1}{3^2-1}+\cdots+\frac{1}{n^2-1}=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2n+2}$$

for every positive integer $n \geq 2$.

Problem 4. Compute the greatest common divisor of the following pairs of positive integers:

(a) 102 and 289

(b) 610 and 987

(c) 1001 and 1729

(d) 888888 and 888888888

Problem 5. Find all pairs of real numbers (x, y) for which $x + y = 2$ and $\frac{x}{y} + \frac{7y}{x} = 8$.

Problem 6. Let H be the orthocenter of triangle ABC such that $AB < BC$. The circumcircle of triangle BHC intersects segment AC again at point X.

(a) Prove that $BA = BX$.

(b) Prove that $HA = HX$.

Problem 7. Let r, s, and t be the three roots of the cubic $3x^3 + 4x^2 - 2$. Find:

(a) rst

(b) $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$

(c) $r^2 + s^2 + t^2$

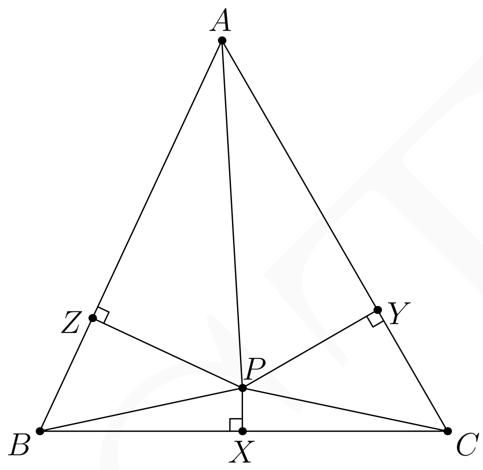
Problem 8. Let $ABCD$ be a cyclic quadrilateral, let X be the orthocenter of triangle ABC, and let Y be the orthocenter of triangle BCD. Prove that quadrilateral $BXYC$ is cyclic.

Problem 9. Determine the incorrect step in the following "proof" that the sum of the first n positive integers is $\frac{1}{2}(n-1)(n+2)$.

> Induct on n. Suppose that the sum of the first k positive integers is $\frac{1}{2}(k-1)(k+2)$. The goal is to prove that the sum of the first $k+1$ positive integers is $\frac{1}{2}((k+1)-1)((k+1)+2)$.
>
> Indeed,
>
> $$\begin{aligned} 1+2+\cdots+k+(k+1) &= (1+2+\cdots+k)+(k+1) \\ &= \tfrac{1}{2}(k-1)(k+2)+(k+1) \\ &= \tfrac{1}{2}(k^2+k-2)+\tfrac{1}{2}(2k+2) \\ &= \tfrac{1}{2}(k^2+3k) \\ &= \tfrac{1}{2}(k)(k+3) \\ &= \tfrac{1}{2}((k+1)-1)((k+1)+2). \end{aligned}$$
>
> The first line follows from the inductive hypothesis, and all remaining lines follow from standard algebraic manipulations. This completes the induction.

Problem 10. Determine the incorrect step in the following "proof" that every triangle is isosceles, and explain why that step is logically incorrect.

Let ABC be a triangle, and let P be the intersection of the angle bisector of $\angle BAC$ and the perpendicular bisector of $\overline{BC}$. Let X, Y, and Z be the feet from P to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$.



Step	Equation	Reasoning
Step 1	$PY = PZ$	P lies on the angle bisector of $\angle BAC$
Step 2	$BP = CP$	P lies on the perpendicular bisector of $\overline{BC}$
Step 3	$BZ = \sqrt{BP^2 - PZ^2}$	Pythagorean theorem
Step 4	$CY = \sqrt{CP^2 - PY^2}$	Pythagorean theorem
Step 5	$BZ = CY$	Substitution using steps 1, 2, 3, and 4
Step 6	$AZ = \sqrt{AP^2 - PZ^2}$	Pythagorean theorem
Step 7	$AY = \sqrt{AP^2 - PY^2}$	Pythagorean theorem
Step 8	$AZ = AY$	Substitution using steps 1, 6, and 7
Step 9	$AB = AC$	Sum the results from steps 5 and 8

Since ABC can be any triangle and $AB = AC$ was deduced, this proves that every triangle is isosceles.