Midterm 2

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Name: ______________________________

Date: ______________________________

This midterm is an exam designed to measure both your progress learning the material and our ability to teach the material. Five of the problems are problems you have already seen in class or on your homework, and five of the problems will be problems you have not seen before.

You will have unlimited time for this exam. Please write up your solutions clearly and concisely. Even for problems that you cannot solve, please write down any partial progress or ideas that you may have. Enjoy the problems!

Problem 0.

(a) How much time did you spend on homework every day outside of classroom time, on average?

(b) Write any final thoughts you have about this mathematics camp.

Problem 1. Determine, with proof, the greatest common divisor of 1971 and 10001.

Problem 2. Prove that the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$.

Problem 3. Find, with proof, all pairs of real numbers (x, y) for which $x + \frac{1}{y} = 2$ and $y + \frac{1}{x} = \frac{9}{4}$.

Problem 4. Let ABC be an acute triangle with circumcenter O such that $AC > BC$. The circumcircle of triangle BOC intersects $\overline{AC}$ again at X. Prove that $AX = BX$.

Problem 5. Prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

for every positive integer n.

Problem 6. Let r and s be distinct real numbers for which $r^2 = 1 + r$ and $s^2 = 1 + s$. Find, with proof, the value of $r^4 + s^4$.

Problem 7. Let F_n denote the n^{th} Fibonacci number. Prove that for all positive integers n, $\gcd(F_n, F_{n+1}) = 1$.

Problem 8. Let $ABCD$ be a cyclic quadrilateral, and let I and J be the incenters of triangles ABC and DBC. Prove that quadrilateral $BIJC$ is cyclic.

Problem 9. Determine the incorrect step in the following "proof" that $1 = 2$, and explain why that step is logically incorrect.

Step	Equation	Reasoning
Step 1	$a = b$	definition
Step 2	$a^2 = ab$	multiply by a
Step 3	$ab + a^2 = 2ab$	add ab
Step 4	$ab - a^2 = 2ab - 2a^2$	subtract $2a^2$
Step 5	$1(ab - a^2) = 2(ab - a^2)$	factor
Step 6	$1 = 2$	cancel $ab - a^2$

Problem 10. Determine the incorrect step in the following "proof" that everyone in Bhutan has the same name, and explain why that step is logically incorrect.

The statement "any group of n people in Bhutan all have the same name" will be proved by induction on n.

(1) The base case $n = 1$ is true, because if there is only one person in the group, then all people in that group have the same name.

(2) For the inductive step, suppose the statement is true for $n = k$, so every group of k people in Bhutan all have the same name. Consider any group of $k + 1$ people; the goal is to prove that everyone in this group of $k + 1$ people has the same name.

(3) First, exclude one person (person A) and look at the remaining k people. By the inductive hypothesis, these k people all have the same name.

(4) Similarly, exclude some other person (person B) and look at the remaining k people. By the same reasoning, these k people all have the same name.

(5) Therefore, person A has the same name as all the $k - 1$ non-excluded people, who all have the same name as person B.

(6) Therefore person A, the $k - 1$ non-excluded people, and person B all have the same name.

(7) Thus everyone in the group of $k + 1$ people has the same name.

Hence, the statement "any group of n people in Bhutan all have the same name" is true. Applying this result to the group of all people in Bhutan shows that all people in Bhutan have the same name.