Midterm 1 Solutions

Ben Kang, Holden Mui, Mark Saengrungkongka

Problem 1. List all the ways to answer a four-question true-false test. How many ways are there?

Solution. The list of ways to answer a four-question true-false test is given below, where "T" denotes "true" and "F" denotes "false."

Question 1	Question 2	Question 3	Question 4
F	F	F	F
F	F	F	T
F	F	T	F
F	F	T	T
F	T	F	F
F	T	F	T
F	T	T	F
F	T	T	T
T	F	F	F
T	F	F	T
T	F	T	F
T	F	T	T
T	T	F	F
T	T	F	T
T	T	T	F
T	T	T	T

In total, there are $\boxed{16}$ ways.

Problem 2. Find, with proof, all real numbers x satisfying

$$2x - \sqrt{3x^2 + 1} = 1.$$

Solution. The only such real number is $x = \boxed{4}$. To see why, let x be a real number satisfying the given equation. Rearranging, squaring, expanding, rearranging, and factoring gives

$$\begin{aligned} 2x - 1 &= \sqrt{3x^2 + 1} \\ (2x-1)^2 &= 3x^2 + 1 \\ 4x^2 - 4x + 1 &= 3x^2 + 1 \\ x^2 - 4x &= 0 \\ x(x-4) &= 0. \end{aligned}$$

Thus if x satisfies the given equation, x must equal 0 or 4.

- If $x = 0$, then x does not satisfy the given equation because

$$2 \cdot 0 - \sqrt{3 \cdot 0^2 + 1} = 0 - \sqrt{1} = -1.$$

- If $x = 4$, then x satisfies the given equation because

$$2 \cdot 4 - \sqrt{3 \cdot 4^2 + 1} = 8 - \sqrt{49} = 1.$$

Therefore, the only real number x satisfying the given equation is $x = 4$.

Problem 3. Find, with proof, the remainder when

$$1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$$

is divided by 9.

Solution A. The sum leaves a remainder of $\boxed{0}$ when divided by 9. To see why, observe that $6! \equiv 720 \equiv 0 \pmod 9$, so the summands $6!$, $7!$, $8!$, $9!$, and $10!$ don't affect the remainder of the sum upon division by 9. Therefore, the given sum leaves a remainder of

$$1! + 2! + 3! + 4! + 5! \equiv 1 + 2 + 6 + 24 + 120 \equiv 153 \equiv 0 \pmod 9$$

when divided by 9.

Solution B. The sum leaves a remainder of $\boxed{0}$ when divided by 9. To see why, computing the factorials by keeping track of the remainder of each factorial modulo 9 gives:

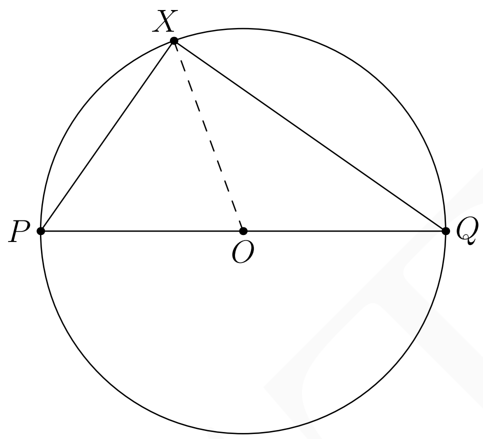
$$\begin{aligned}
1! &\equiv 1 && \pmod 9 \\
2! &\equiv 2 \cdot 1! \equiv 2 \cdot 1 \equiv 2 && \pmod 9 \\
3! &\equiv 3 \cdot 2! \equiv 3 \cdot 2 \equiv 6 && \pmod 9 \\
4! &\equiv 4 \cdot 3! \equiv 4 \cdot 6 \equiv 24 \equiv 6 && \pmod 9 \\
5! &\equiv 5 \cdot 4! \equiv 5 \cdot 6 \equiv 30 \equiv 3 && \pmod 9 \\
6! &\equiv 6 \cdot 5! \equiv 6 \cdot 3 \equiv 18 \equiv 0 && \pmod 9
\end{aligned}$$

Since $6!$ is divisible by 9, the summands $7!$, $8!$, $9!$, and $10!$ are all also divisible by 9. Therefore, the given sum leaves a remainder of

$$1! + 2! + 3! + 4! + 5! \equiv 1 + 2 + 6 + 6 + 3 \equiv 18 \equiv 0 \pmod 9$$

when divided by 9.

Problem 4. Let Ω be a circle with diameter $\overline{PQ}$, and let X be a point on Ω different from P and Q. Prove that $\overline{PX}$ is perpendicular to $\overline{QX}$.



Solution A. Let O be the center of Ω. Since $\overline{PQ}$ is a diameter, O lies on $\overline{PQ}$, so $\angle POQ = 180°$. By the inscribed angle theorem, $\angle POQ = 2\angle PXQ$. Therefore, $\angle PXQ = \frac{1}{2} \cdot 180° = 90°$.

Solution B. Let O be the center of Ω, and let $\angle OQX = \theta$.

- Since $OX = OQ$, $\angle OXQ = \theta$.
- Since the angles in triangle OQX sum to $180°$, $\angle QOX = 180° - 2\theta$.
- Since $\angle POQ$ is a straight angle, $\angle POX = 2\theta$.
- Since $OP = OX$, $\angle PXO = \frac{180° - 2\theta}{2} = 90° - \theta$.

Therefore $\angle PXQ = (90° - \theta) + \theta = 90°$.

Solution C. Let O be the center of Ω. Since $\overline{PQ}$ is a diameter, O lies on $\overline{PQ}$. Since $OP = OX$, $\angle OPX = \angle OXP = \alpha$ for some angle α. Similarly, $\angle OQX = \angle OXQ = \beta$ for some angle β. Since the angles in a triangle sum to $180°$, $\angle POX = 180° - 2\alpha$ and $\angle QOX = 180° - 2\beta$. Since these angles form a straight angle,

$$(180° - 2\alpha) + (180° - 2\beta) = 180°,$$

which simplifies to $\alpha + \beta = 90°$. Therefore $\angle PXQ = \alpha + \beta = 90°$.

Problem 5. Jigme and Dorji are playing a game on a chocolate bar divided into a 3×5 grid of chocolate pieces. Jigme goes first, and they alternate breaking one of the chocolate chunks into exactly two smaller pieces along the grid of the chocolate bar (the break line can be zigzag-shaped as long as it follows the grid lines). The last player to make a move wins. Find, with proof, the player with the winning strategy.

Solution. $\boxed{\text{Dorji}}$ has the winning strategy. In fact, Dorji always wins regardless of what moves are made. To see why, observe that

- the game starts with one chocolate chunk,
- the game ends with 15 chocolate chunks, and
- each move increases the number of pieces by exactly one.

Therefore, every game ends in exactly 14 moves. This is an even number, so the second player, Dorji, always makes the last move.

Problem 6. Call a number *threeven* if it is of the form $3k - 1$ for some integer k, and call a number *throdd* if it is of the form $3k - 2$ for some integer k.

(a) Is the product of two threeven numbers always threeven?

(b) Is the product of two throdd numbers always throdd?

Solution A.

(a) The product of two threeven numbers is not always threeven. To see why, observe that 2 is threeven because $3 \cdot 1 - 1 = 2$. However, 4 is not threeven because $3k - 1 = 4$ has no integer solutions. Since $2 \cdot 2 = 4$, the product of two threeven numbers is not neccessarily threeven.

(b) The product of two throdd numbers is always odd. To see why, let $3a-2$ and $3b-2$ be two threeven numbers, where a and b are integers. Since

$$(3a - 2)(3b - 2) = 9ab - 6a - 6b + 4 = 3(3ab - 2a - 2b + 2) - 2$$

and $3ab - 2a - 2b + 2$ is an integer, $(3a - 2)(3b - 2)$ must be throdd.

Solution B.

(a) The product of two threeven numbers is not always threeven. In fact, the product of two threeven numbers is never threeven. To see why, observe that threeven numbers are exactly the integers congruent to -1 modulo 3. Since

$$(-1) \cdot (-1) \equiv 1 \not\equiv -1 \pmod 3,$$

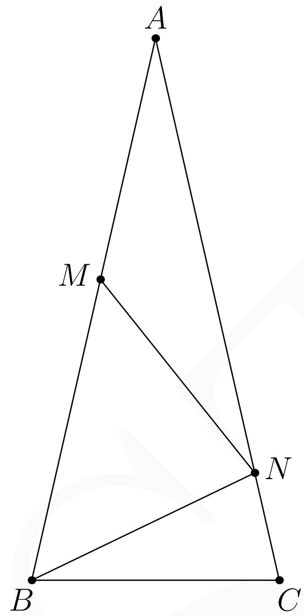
the product of two threeven numbers is never threeven.

(b) The product of two throdd numbers is always throdd. To see why, observe that throdd numbers are exactly the integers congruent to -2 modulo 3. Since

$$(-2) \cdot (-2) \equiv 4 \equiv -2 \pmod 3,$$

the product of two throdd numbers is always throdd.

Problem 7. Let ABC be a triangle with $AB = AC$. Point M lies on side $\overline{AB}$ and point N lies on side $\overline{AC}$ such that $AM = MN = NB = BC$. Find, with proof, the measure of $\angle BAC$.



Solution. The answer is $\angle BAC = \boxed{\frac{180^\circ}{7}}$. To see why, let $\angle BAC = x$.

- Since $MA = MN$, $\angle MNA = x$.
- Since the angles in triangle AMN sum to 180°, $\angle AMN = 180^\circ - 2x$.
- Since $\angle AMB$ is a straight angle, $\angle BMN = 2x$.
- Since $BN = MN$, $\angle MBN = 2x$.
- Since the angles in triangle ABN sum to 180°, $\angle ANB = 180^\circ - 3x$.
- Since $\angle ANC$ is a straight angle, $\angle BNC = 3x$.
- Since $BC = BN$, $\angle BCN = 3x$.
- Since $AB = AC$, $\angle ABC = 3x$.

Since the angles in triangle ABC sum to 180°, $x + 3x + 3x = 180^\circ$. Solving for x gives $x = \frac{180^\circ}{7}$.

Problem 8. How many ways are there to arrange the letters in "BHUTAN" such that A and B are adjacent, and T and U are adjacent?

Solution A. There are $\boxed{96}$ arrangements. To see why, observe there are

- $4!$ ways to arrange the letters A, T, H, and N,
- 2 ways to place the B since it can be directly in front of the A or directly behind the A, and
- 2 ways to place the U since it can be directly in front of the T or directly behind the T.

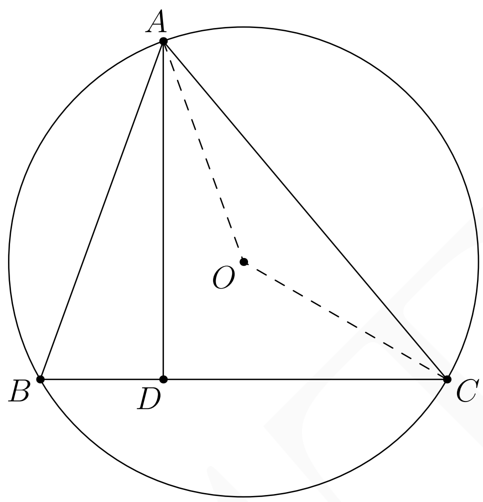
By the counting principle, there are $4! \cdot 2 \cdot 2 = 96$ arrangements.

Solution B. There are $\boxed{96}$ arrangements. To see why, group AB together and group TU together. There are:

- $4!$ ways to arrange AB, TU, H, and N
- 2 ways to arrange the letters within the "AB" group, and
- 2 ways to arrange the letters within the "TU" group.

By the counting principle, there are $4! \cdot 2 \cdot 2 = 96$ arrangements.

Problem 9. Let ABC be an acute triangle with circumcenter O, and let AD be an altitude of triangle ABC. Prove that $\angle BAD = \angle OAC$.



Solution A. Let $\angle BAD = \theta$. Since $\angle ADB$ is a right angle, $\angle ABD = 90^\circ - \theta$. By the inscribed angle theorem, $\angle AOC = 2 \cdot (90^\circ - \theta)$. Since $AO = OC$,

$$\angle OAC = \frac{180^\circ - 2 \cdot (90^\circ - \theta)}{2} = \frac{180^\circ - 180^\circ + 2\theta}{2} = \theta,$$

as desired.

Solution B. Let $\angle OAC = \theta$. Since $OA = OC$, $\angle OCA = \theta$. Since the angles in triangle AOC sum to 180°, $\angle AOC = 180^\circ - 2\theta$. By the inscribed angle theorem,

$$\angle ABC = \frac{180^\circ - 2\theta}{2} = 90^\circ - \theta.$$

Since $\angle ADB$ is a right angle, $\angle BAD = 90^\circ - (90^\circ - \theta) = \theta$, as desired.

Problem 10. Let x be a real number satisfying $2^x = 3$. Prove that x is irrational.

Solution. Assume for contradiction that x is rational. Since $2^x \leq 1$ when x is not positive, x must be positive and hence equals $\frac{a}{b}$ for some positive integers a and b. Because $2^{a/b} = 3$, raising both sides to the power of b gives

$$2^a = 3^b.$$

Since a is a positive integer, 2^a must be even. However, 3^b is odd. Hence 2^a cannot be equal to 3^b, which contradicts their equality.