Midterm 1 Review

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Name: ______________________________

Date: _______________________________

The first midterm exam will be held at 9:30am on Saturday, January 20, 2024. No calculators or notes are allowed during the exam. It is designed to measure both your progress learning the material and our ability to teach the material. Five of the problems will be problems you have already seen in class or on your homework, and five of the problems will be problems you have not seen before. You will have unlimited time for the exam.

The problems on this handout are intended to help you review for the exam. These problems will be similar in difficulty, style, and topic to the midterm exam problems.

Problem 1. There is a pile of 20 stones, and Pema and Yeshi alternate taking either 1 or 2 stones from the pile at a time. Pema goes first, and a player wins if they take the last stone. Who has the winning strategy?

Problem 2. List the following:

(a) all three-digit numbers for which the sum of the first two digits equals the last digit

(b) all ways to answer a five-question true-false exam such that at least half of the answers are "true"

Problem 3. Find all real numbers x for which:

(a) $\sqrt{2x^2 - 7} - x = 1$

(b) $\sqrt{x} - \sqrt{2x - 1} = 2$

Problem 4.

(a) Find the remainder when

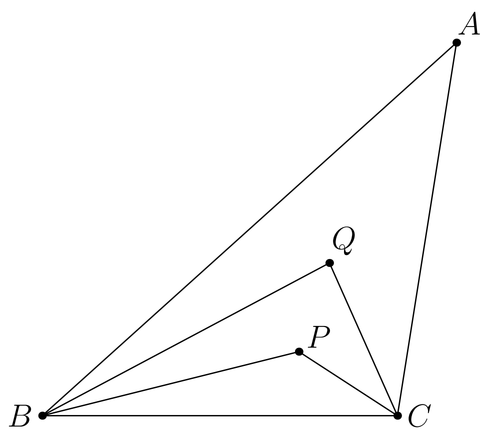
$$18 \times 81 \times 108 \times 180 \times 801 \times 810$$

is divided by 7.

(b) Find the last two digits of 21^{108}.

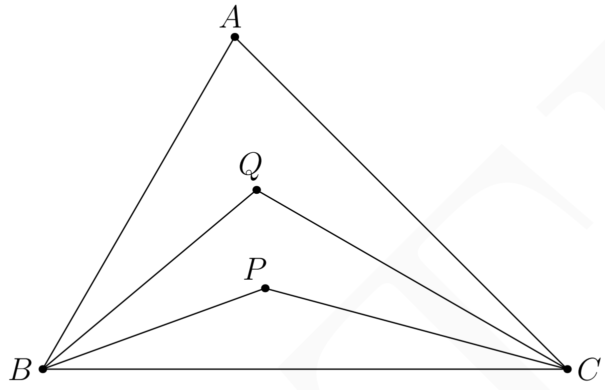
Problem 5.

(a) In triangle ABC, the angle trisectors of $\angle B$ and $\angle C$ meet at points P and Q in the diagram below. Given that $\angle A = 39^\circ$ and $\angle QBP = 14^\circ$, find $\angle BPC$.



(b) In triangle ABC, the angle trisectors of $\angle B$ and $\angle C$ meet at points P and Q such that P lies inside triangle BCQ. Prove that

$$\angle BQC = \tfrac{1}{2}(\angle BAC + \angle BPC).$$



Problem 6. Simplify:

(a) $\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}$

(b) $\sqrt{2+\sqrt{3}} - \sqrt{2-\sqrt{3}}$

(c) $\sqrt{2+\sqrt{3}}$

Problem 7. Call a positive integer *composite* if it can be written in the form ab, where a and b are positive integers greater than one.

(a) Is the sum of two composite numbers always composite?

(b) Is the product of two composite numbers always composite?

Problem 8. Let Ω be a circle with center O and let $ABCDEF$ be a regular hexagon whose vertices lie on Ω. Point P lies on minor arc AB such that $\angle BAP = 20^\circ$.

(a) Find $\angle BOP$.

(b) Find $\angle CPO$.

Problem 9. Find the number of ways to arrange the letters in THIMPHU such that:

(a) there are exactly four letters between the H's

(b) there is exactly one letter between the H's

Problem 10. Let a, b, and c be positive integers such that $a^2 + b^2 = c^2$.

(a) Prove that abc is a multiple of 3.

(b) Prove that abc is a multiple of 4.

(c) Prove that abc is a multiple of 5.