Algebraic Manipulations

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> Many problems in mathematical competitions concern algebraic expressions. Experienced problem solvers approach these problems with extensive knowledge of common identities and skillful manipulations of expressions.

Problem 1. Explain whether or not the following manipulations are valid. If not valid, give a numerical counterexample.

(a) $x^2 + y^2 = (x + y)^2$

(b) $x(y - z) = xy - xz$

(c) $\frac{x}{\frac{y}{z}} = \frac{xz}{y}$ for nonzero y and z

(d) $(xy)^2 = x^2y^2$

(e) $2xy = x^2y^2$

(f) $x - (y + z) = x - y + z$

(g) $(x - y)(x + y) = x^2 - y^2$

(h) $\dfrac{x^3 + y^3}{x + y} = x^2 + y^2$

(i) $\dfrac{x^3}{x + y + z} = \dfrac{x^2}{y + z}$

(j) $\dfrac{x + y}{x + z} = \dfrac{y}{z}$

Problem 2. Expand the following expressions:

(a) $(x + 7)(x + 11)$

(b) $(x - 7)(x + 11)$

(c) $(2x - 7)(x - 11)$

(d) $(x^2 - x + 2)(x - 4)$

(e) $(3x^2 + 5)(x^2 - 3x + 1)$

(f) $(x - 1)(x + 2)(2x - 3)$

(g) $(3x^2 - x - 2)(x^2 - 2x - 3)$

Problem 3. Factor the following polynomials:

(a) $x^2 - 5x + 4$

(b) $x^2 + 12x + 35$

(c) $x^2 + 6x + 9$

(d) $x^2 - 2x - 15$

(e) $x^2 + 5x - 24$

(f) $2x^2 + 5x + 3$

(g) $2x^2 - 7x + 3$

Problem 4. Expand the following expressions:

(a) $(a+b)^2$

(b) $(a-b)^2$

(c) $(a-b)(a+b)$

(d) $(a+b)^3$

(e) $(a-b)^3$

(f) $(a+b)(a^2-ab+b^2)$

(g) $(a-b)(a^2+ab+b^2)$

Problem 5. Expand the following expressions:

(a) $(a+b+c)(x+y+z)$

(b) $(a+b+c)^2$

(c) $(a-b+c)(x-y+z)$

(d) $(a-b+c)^2$

(e) $(a+b)(m+n)(x+y)$

(f) $(a+1)(b+1)(c+1)$

(g) $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

Problem 6. Prove the following identities:

(a) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(b) $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

(c) $(x+y+z)(xy+yz+zx) = (x+y)(y+z)(z+x) + xyz$

The next few problems concern applications of common identities.

Problem 7. Let a and b be real numbers such that $a+b=10$ and $ab=7$.

(a) Compute a^2+b^2.

(b) Compute a^3+b^3.

Problem 8. Find all real numbers x for which:

(a) $2x - \sqrt{3x^2 + 1} = 1$

(b) $\sqrt{2x^2 + 1} - x = 5$

(c) $\sqrt{x + 5} - \sqrt{x} = 2$

(d) $\sqrt{x + 2} + \sqrt{x} = 5$

Problem 9. Let x be a real number such that $\sqrt{x+2}+\sqrt{x}=5$.

(a) Without solving for x, compute $\sqrt{x+2}-\sqrt{x}$.

(b) Using part (a), solve for x.

(c) Using a similar method from part (a) and (b), find all real numbers y for which $\sqrt{3y-2}-\sqrt{y-2}=y$.

Problem 10. Heron's formula states that the area of a triangle with side lengths a, b, c is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s=\frac{1}{2}(a+b+c)$. Using this formula, compute the area of a triangle with side lengths $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$.

The following problems are intended to demonstrate useful algebraic identities. You may encounter these again in the future.

Problem 11.

(a) Prove that $a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

(b) State a similar identity for $a^n - b^n$.

(c) Prove that $a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

(d) Prove that $2^{35} - 1$ is a composite number.[1]

[1] If n is composite, then $2^n - 1$ is composite. Thus if $2^n - 1$ is prime, then n must be prime. Prime numbers of the form $2^n - 1$ form are called *Mersenne primes*. The largest known prime, $2^{82589933} - 1$, is of this form.

Problem 12 (Sophie Germain's identity).

(a) Fill in the blank with the correct expression to get factorization of $a^4 + 4b^4$.

$$a^4 + 4b^4 = (a^4 + \underline{\hspace{2cm}} + 4b^4) - \underline{\hspace{2cm}}$$

$$= (a^2 + \underline{\hspace{2cm}})^2 - \underline{\hspace{2cm}}$$

$$= (a^2 + \underline{\hspace{2cm}})^2 - (\underline{\hspace{2cm}})^2$$

$$= (\underline{\hspace{5cm}})(\underline{\hspace{5cm}}).$$

(b) Using a similar manipulation, factor $x^4 + x^2 + 1$.

Problem 13.

(a) Fill in the blank to form a correct identity.

$$(a^2 + b^2)(c^2 + d^2) = (\underline{\hspace{2cm}} + \underline{\hspace{2cm}})^2 + (\underline{\hspace{2cm}} - \underline{\hspace{2cm}})^2.$$

(b) An integer is called *great* if it can be written as a sum of two perfect squares. Prove that the product of any two great integers is also great.