Journey to the International Math Olympiad

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> Here are four IMO problems whose solutions use techniques you have learned about at this camp. Even though you will be able to understand the solutions to these problems, you will also notice coming up with such a solution is no easy task.
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> Take a moment to reflect on where you were at the beginning of the camp, where you are now, and what it takes to get to where you want to be in the future. This camp is the first step to success at the IMO. The journey to the IMO is a long journey, but with persistence, hard work, and a willingness to learn, anything is possible.

Problem 1 (IMO 2014, Problem 1). Let $a_0 < a_1 < a_2 < \dots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

Problem 2 (IMO 2011, Problem 2). Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Problem 3 (IMO 2005, Problem 4). Determine all positive integers relatively prime to all the terms of the infinite sequence $a_1, a_2, a_3, \ldots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1$$

for all positive integers n.

Problem 4 (IMO 2006, Problem 1). Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.