Induction

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Mathematical *induction* is a way of proving a statement for all positive integers n. Almost every induction proof consists of two steps:

(a) proving that the statement is true for $n = 1$, and

(b) proving that the truth of the statement for $n = k$ implies the truth of the statement for $n = k + 1$.

The first step is known as the *base case* and the second step is known as the *inductive step*. There are many types of induction proofs, but all of them take this general form.

Problem 1. Prove the following sums hold for all positive integers n:

(a) $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

(b) $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(c) $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

(d) $1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{1}{2}(3^n - 1)$

(e) $1 + r + r^2 + \cdots + r^{n-1} = \dfrac{r^n - 1}{r - 1}$ for all real numbers $r \neq 1$

Problem 2. Conjecture a closed form for the following sums by substituting small values of n. Then, use induction to prove the conjecture.

(a) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n \cdot (n+1)}$

(b) $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{(2n-1) \cdot (2n+1)}$

Problem 3. Conjecture a closed form for the following sequences by computing the first few terms of each sequence. Then, use induction to prove the conjecture.

(a) A sequence with starting term $a_1 = 1$ and recurrence $a_n = 2a_{n-1} + 1$ for all integers $n > 1$

(b) A sequence with starting term $a_1 = 1$ and recurrence $a_n = \frac{a_{n-1}}{a_{n-1}+1}$ for all integers $n > 1$

Problem 4. Find all positive integers n for which:

(a) $2^n < n^2$

(b) $2^{n-2} < n^2$

(c) $n! < 3^n$

(d) $n! < 4^n$

Problem 5. Let n be a positive integer. Prove that

$$(1+x)^n \geq 1+nx$$

for any real number $x \geq -1$.

Problem 6. The *Fibonacci sequence* is the sequence with initial terms $F_1 = F_2 = 1$ and recurrence $F_n = F_{n-1} + F_{n-2}$ for all positive integers $n > 2$.

(a) Prove that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

(b) Prove that

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$

Problem 7. Prove that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$$

for every positive integer n.

Problem 8. Prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \leq 2$$

for every positive integer n.

The following problems demonstrate how induction can be used in number theory.

Problem 9. Prove that for all positive integers n:

(a) 3^{n+1} is a factor of $2^{3^n} + 1$

(b) 5^{n+1} is a factor of $4^{5^n} + 1$

(c) 2^{n+2} is a factor of $3^{2^n} - 1$

Problem 10.

(a) Prove that the product of any two consecutive positive integers is divisible by 2.

(b) Prove that the product of any three consecutive positive integers is divisible by 6.

(c) Prove that the product of any four consecutive positive integers is divisible by 24.

(d) Prove that the product of any k consecutive positive integers is divisible by $k!$.

Problem 11.

(a) Prove that for every positive integer n, there exists an n-digit number divisible by 5^n whose digits are all odd.

(b) Prove that for every positive integer n, there exists an n-digit number divisible by 2^n whose digits are all 1 or 2.

The following problems demonstrate how induction can be used to prove statements in combinatorics.

Problem 12. Let F_n denote the Fibonacci sequence.

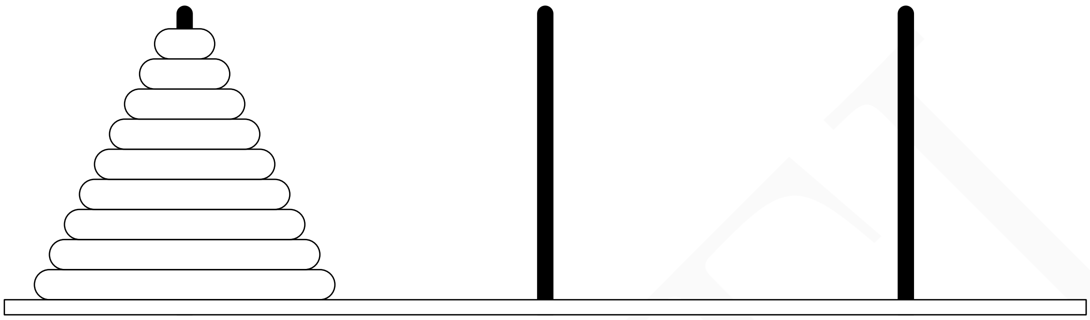
(a) Prove that the number of ways to tile a $2 \times n$ grid with 1×2 dominoes is F_{n+1}.

(b) Prove that the number of ways to answer an n-question true-false test such that no two consecutive questions are answered "false" is F_{n+2}.

(c) Prove that the number of ways to write n as an ordered sum of positive integers is 2^{n-1}.

(d) Prove that the number of ways to write n as an ordered sum of odd positive integers is F_n.

Problem 13 (Tower of Hanoi). Let n be a positive integer. There are three posts and a stack of n disks, initially placed on the leftmost post in order of increasing size.

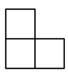


A valid move consists of taking any top disk on any post and moving it to another post such that it is not placed on top of a disk with smaller size. The goal is to move all the disks to the rightmost post.

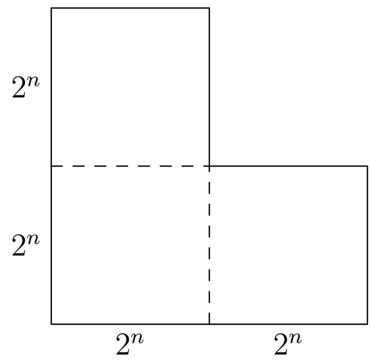
(a) List out a sequence of moves that achieves this goal for $n = 4$.

(b) Prove that the goal is always possible for all positive integers n.

Problem 14. An *L-tromino* is a shape formed by 3 squares as shown below. It may be rotated and reflected.



(a) Prove that for any positive integer n, the figure below (obtained by connecting three $2^n \times 2^n$ squares) can be tiled with L-trominoes.



(b) Prove that for any positive integer n, if one cell is removed from a $2^n \times 2^n$ grid, the remaining cells can be tiled with L-trominoes.

(c) Use the proof from parts (a) and (b) to tile the 255 squares below with L-trominoes.