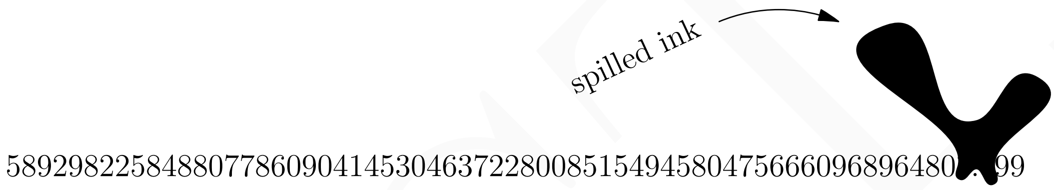
# Final Exam Solutions

Ben Kang, Holden Mui, Mark Saengrungkongka

**Problem 1.** Jigme is writing the decimal expansion of $(2+\sqrt{3})^{108}$ with a pen. Unfortunately, her pen breaks in half after she writes the first 65 digits. The pen ink spills on her paper, covering the ones digit and the tenths digit of the number. Her paper looks like this:




The goal of this problem is to help Jigme determine the covered digits.

(a) Prove that

$$x^{n+1}+y^{n+1}=(x+y)(x^n+y^n)-(xy)(x^{n-1}+y^{n-1})$$

for every positive integer $n$.

(b) Let $s_n=(2+\sqrt{3})^n+(2-\sqrt{3})^n$. Using the identity from part (a), prove that $s_1=4$, $s_2=14$, and $s_{n+1}=4s_n-s_{n-1}$ for all integers $n\geq 2$.

(c) Use the recurrence from part (b) to prove that $s_{108}$ is an integer. Then, prove that $2-\sqrt{3}$ is between 0 and 0.5. Finally, use both facts to determine the tenths digit of $(2+\sqrt{3})^{108}$.

(d) Use the recurrence from part (b) to determine the sequence $s_1, s_2, \ldots$ modulo 10. Then, use your result to find the units digit of $(2+\sqrt{3})^{108}$.

*Solution.*

(a) By expansion,

$$\begin{aligned}&(x+y)(x^n+y^n)-(xy)(x^{n-1}+y^{n-1})\\&=x^{n+1}+xy^n+x^ny+y^{n+1}-x^ny-xy^n\\&=x^{n+1}+y^{n+1}.\end{aligned}$$

(b) Computing the values of $s_1$ and $s_2$ gives

$$s_1=(2+\sqrt{3})+(2-\sqrt{3})=4$$

and

$$s_2=(2+\sqrt{3})^2+(2-\sqrt{3})^2=(7+4\sqrt{3})+(7-4\sqrt{3})=14.$$

To prove that $s_{n+1}=4s_n-s_{n-1}$ for all integers $n\geq 2$, use part (a) to get

$$\begin{aligned}s_{n+1}&=(2+\sqrt{3})^{n+1}+(2-\sqrt{3})^{n+1}\\&=\left[(2+\sqrt{3})+(2-\sqrt{3})\right]\left[(2+\sqrt{3})^n+(2-\sqrt{3})^n\right]\\&\qquad-\left[(2+\sqrt{3})(2-\sqrt{3})\right]\left[(2+\sqrt{3})^{n-1}+(2-\sqrt{3})^{n-1}\right]\\&=4\left[(2+\sqrt{3})^n+(2-\sqrt{3})^n\right]-\left[(2+\sqrt{3})^{n-1}+(2-\sqrt{3})^{n-1}\right]\\&=4s_n-s_{n-1}.\end{aligned}$$

(c) All terms in the sequence $s_1, s_2, \ldots$ are integers because

- the recurrence $s_{n+1}=4s_n-s_{n-1}$ shows that each term in the sequence is an integer if the previous two terms are integers, and
- the first two terms of the sequence, $s_1=4$ and $s_2=14$, are integers.

Therefore, $s_{108}$ is an integer.

To prove that $2-\sqrt{3}$ is between 0 and 0.5, observe that $\sqrt{3}$ is between 1.5 and 2 because $1.5^2=2.25<3<4=2^2$. Therefore, $0<2-\sqrt{3}<0.5$.

To find the tenths digit of $(2+\sqrt{3})^{108}$, first observe that

$$(2+\sqrt{3})^{108}=s_{108}-(2-\sqrt{3})^{108}.$$

Since $2-\sqrt{3}$ is between 0 and 0.5, raising it to the power of 108 will make it a very small positive number. Since $s_{108}$ is a positive integer, $(2+\sqrt{3})^{108}$ must be slightly less than an integer, so its tenths digit is $\boxed{9}$.

(d) Computing the first five terms of the sequence gives $s_1 = 4$, $s_2 = 14$, $s_3 = 52$, $s_4 = 194$, and $s_5 = 724$. Modulo 10, this sequence becomes 4, 4, 2, 4, 4. Since each term of the sequence only depends on the previous two terms, the sequence of remainders modulo 10 must repeat every three terms to give the sequence $4, 4, 2, 4, 4, 2, 4, 4, 2, \ldots \pmod{10}$.

Since $108 \equiv 0 \pmod 3$, the last digit of $s_{108}$ must equal the last digit of $s_3$, which is 2. Since

$$(2+\sqrt{3})^{108} = s_{108} - (2-\sqrt{3})^{108}$$

and $2-\sqrt{3}$ is between 0 and 1, the units digit of $(2+\sqrt{3})^{108}$ must be $\boxed{1}$.

**Problem 2.** Tandin, Tashi, Tenzin, Thinley, Tshering, and Tshewang are at a party, and each pair of people are either friends or strangers.

Call a group of people *mutual friends* if every pair of them are friends, and call a group of people *mutual strangers* if every pair of them are strangers. The goal of parts (a) and (b) is to prove that there is a group of three people at the party that are either mutual friends or mutual strangers.

(a) Prove that one of the two statements below must be true:

- There are three people at the party that are each friends with Tandin.
- There are three people at the party that are each strangers with Tandin.

(b) Consider the three people from part (a), along with Tandin. Prove that among these four people, three of them must be mutual friends or mutual strangers.

In fact, there must be *two* groups of three people that are mutual friends or mutual strangers, regardless of who are friends and who are strangers. The goal of parts (c) and (d) is to prove this statement.

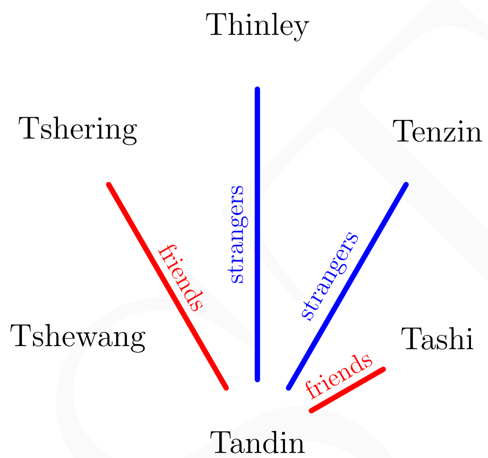
(c) Explain why proving this statement in the case that Tandin, Tashi, and Tenzin are mutual strangers implies the statement is true in all possible cases.

(d) Assuming that Tandin, Tashi, and Tenzin are mutual strangers, prove that some other group of three people are mutual friends or mutual strangers in the case that:

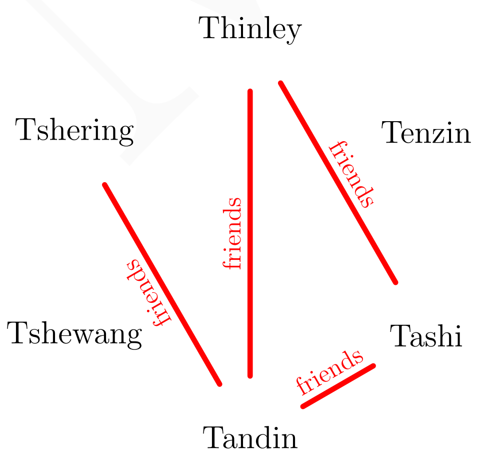
- no two people among Thinley, Tshering, and Tshewang are friends.
- some two people among Thinley, Tshering, and Tshewang are friends.

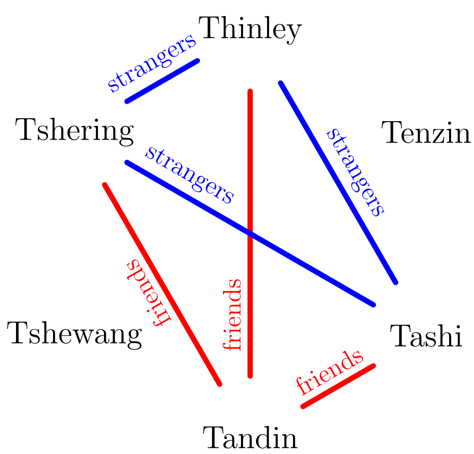
*Solution.*

(a) Suppose for contradiction that neither statement is true. Then Tandin must be friends with at most two people, and Tandin must be strangers with at most two people. Therefore, there are at most four other people at the party, which is a contradiction because there are five other people at the party.

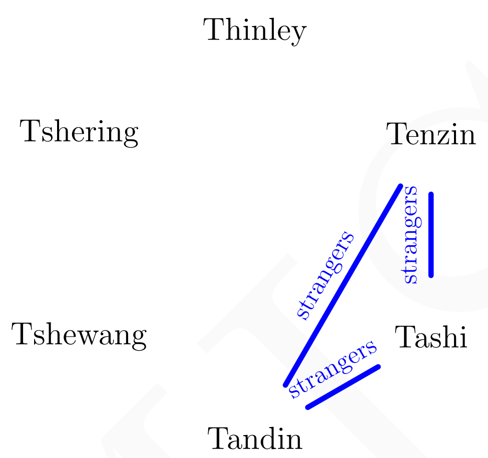


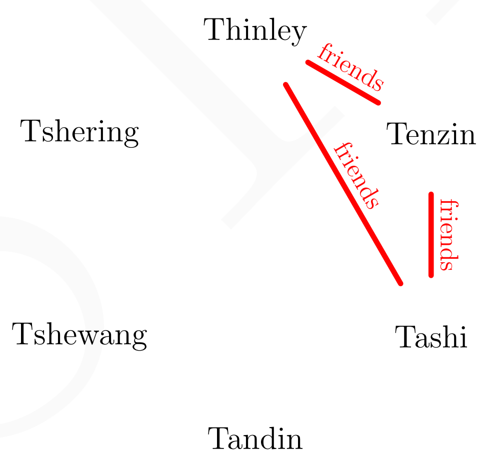

(b) If the friendship status between any two of the three people from part (a) is the same as the friendship status between Tandin and the three people, then those two people and Tandin are either mutual friends or mutual strangers. Otherwise, the friendship status between every pair of people from part (a) is the opposite of the friendship status between Tandin and the three people, so these three people must be mutual friends or mutual strangers.



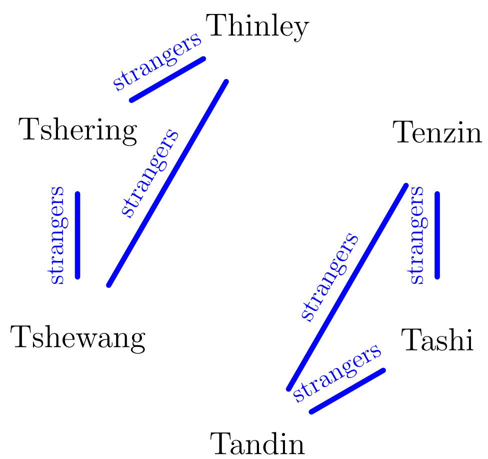



(c) Suppose that there is a proof in the case that Tandin, Tashi, and Tenzin are mutual strangers. By part (b), some three people are mutual friends or mutual strangers. If these three people are mutual strangers, then the proof in the case that Tandin, Tashi, and Tenzin are mutual strangers can be repeated by replacing their names with the names of the three people that are mutual strangers. Similarly, if these three people are mutual friends, then the proof in the case that Tandin, Tashi, and Tenzin are mutual strangers can be repeated by replacing their names with the names of the three people that are mutual friends, and also replacing every instance of "mutual strangers" with "mutual friends" and vice versa.

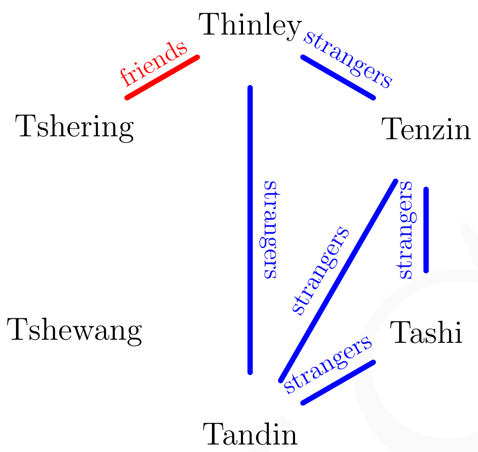


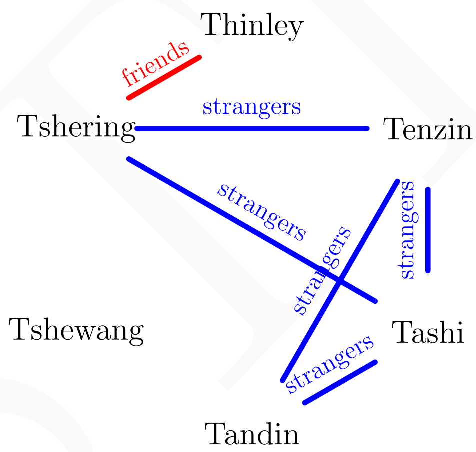




(d) If no two people among Thinley, Tshering, and Tshewang are friends, then these three people are mutual strangers.

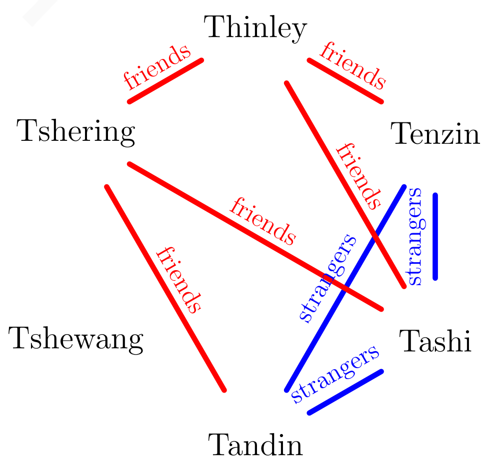


If two people among Thinley, Tshering, and Tshewang are friends, then assume that Thinley and Tshering are friends using the same reasoning as part (c). If Thinley is strangers with two of Tandin, Tashi, and Tenzin, then Thinley and these two people are mutual strangers. Similarly, if Tshering is strangers with two of Tandin, Tashi, and Tenzin, then Tshering and these two people are mutual strangers.




If neither of these are true, then Thinley is friends with at least two of Tandin, Tashi, and Tenzin, and Tshering is friends with at least two of Tandin, Tashi, and Tenzin. Thus, one of Tandin, Tashi, and Tenzin is friends with both Thinley and Tshering. This person, along with Thinley and Tshering, must be mutual friends.



**Problem 3.** Gyeltshen encounters the following geometry problem on the Internet.

> Let $ABC$ be an acute triangle with circumcenter $O$, and let point $D$ lie on side $\overline{BC}$. Point $E$ lies on $\overline{AC}$ such that $\overline{DE} \perp \overline{CO}$, and point $F$ lies on $\overline{AB}$ such that $\overline{DF} \perp \overline{BO}$. Let $K$ be the circumcenter of triangle $AEF$. Prove that $\overline{DK}$ and $\overline{BC}$ are perpendicular.

Gyeltshen starts solving the problem by setting $\angle BAC = \alpha$, but he gets stuck because his triangle looks equilateral and his diagram is both small and inaccurate.

(a) Help Gyeltshen draw a large and accurate diagram. Make sure that triangle $ABC$ is acute and scalene. Then, express both $\angle BOC$ and $\angle EKF$ in terms of $\alpha$.

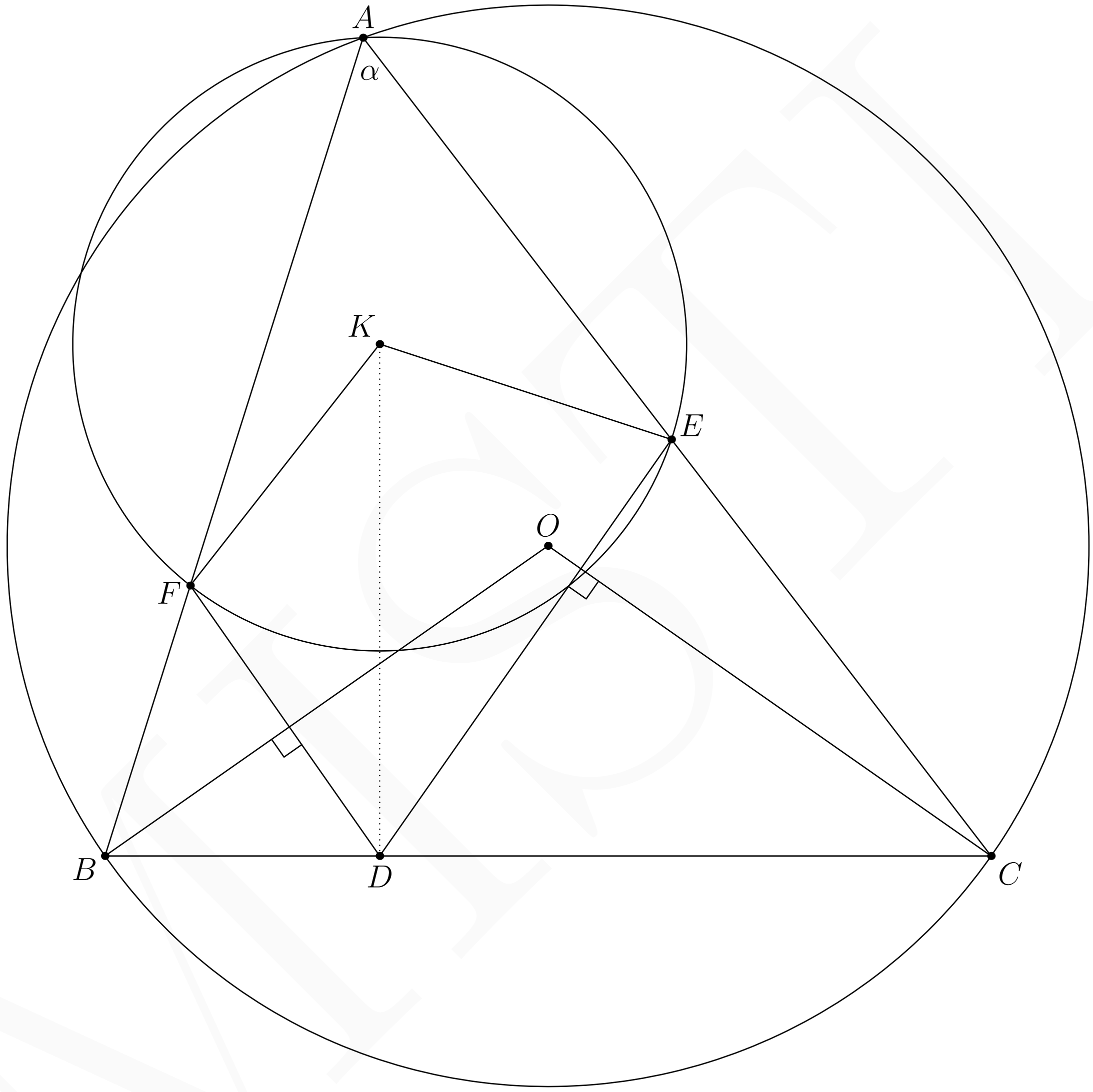
(b) Gyeltshen, with the help of your large and accurate diagram, conjectures that $\angle BDF = \alpha$ and $\angle CDE = \alpha$. Using the result from part (a), help Gyeltshen prove this conjecture.

(c) Gyeltshen also notices that $D$, $E$, $F$, and $K$ appear to be concyclic. Using the result from part (b), help Gyeltshen prove that $D$, $E$, $F$, and $K$ are concyclic.

(d) Gyeltshen, using the result from part (c), draws a circle through points $D$, $E$, $F$, and $K$. Using this cyclic quadrilateral, prove that $\overline{DK}$ is perpendicular to $\overline{BC}$.

*Solution.*

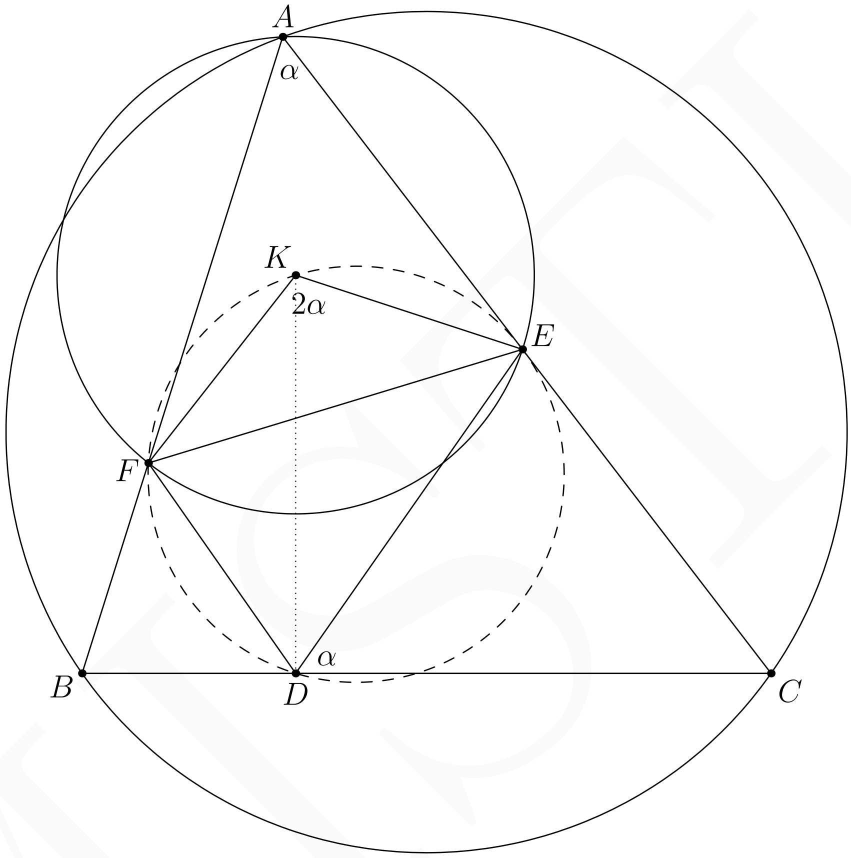
(a) A large, accurate diagram is given below.




Since $O$ is the circumcenter of triangle $ABC$, $\angle BOC = \boxed{2\alpha}$ by the inscribed angle theorem. Similarly, $\angle EKF = \boxed{2\alpha}$ because $K$ is the circumcenter of triangle $AEF$.

(b) Since $OB = OC$, both $\angle OBC$ and $\angle OCB$ have a degree measure of $90^\circ - \alpha$. Since the triangle formed by lines $\overline{DF}$, $\overline{BD}$ and $\overline{BO}$ is a right triangle, $\angle BDF = \alpha$. Similarly, the triangle formed by lines $\overline{DE}$, $\overline{CD}$, and $\overline{CO}$ is a right triangle, so $\angle CDE = \alpha$.

(c) Since $\angle BDC$ is a straight angle, $\angle EDF = 180^\circ - 2\alpha$. Additionally, $\angle EKF = 2\alpha$ by part (a). Therefore, $\angle EDF + \angle EKF = 180^\circ$, so quadrilateral $DEKF$ is cyclic.




(d) Since $EK = FK$, $\angle EFK = 90^\circ - \alpha$. Next, applying the inscribed angle theorem on the circle from part (c) gives $\angle EDK = 90^\circ - \alpha$. Using the fact that $\angle CDE = \alpha$ from part (b) gives

$$\angle KDC = (90^\circ - \alpha) + \alpha = 90^\circ,$$

implying that $\overline{DK}$ is perpendicular to $\overline{BC}$.

**Problem 4.** Yeshi thinks that every prime number can be expressed as the sum or the difference of a power of 2 and a power of 3, in some order. Pema thinks that Yeshi is wrong. The goal of this problem is to determine who is right.

(a) Yeshi is able to write the first five prime numbers as a sum or a difference of a power of 2 and a power of 3. However, Pema is not convinced. Help Yeshi write the next ten prime numbers as the sum or the difference of a power of 2 and a power of 3.

$$\begin{aligned}
2 &= 2^0 + 3^0 \\
3 &= 2^1 + 3^0 \\
5 &= 2^1 + 3^1 \\
7 &= 2^3 - 3^0 \\
11 &= 3^3 - 2^4 \\
13 &= \\
17 &= \\
19 &= \\
23 &= \\
29 &= \\
31 &= \\
37 &= \\
41 &= \\
43 &= \\
47 &=
\end{aligned}$$

Pema is still not convinced that every prime number can be expressed as a sum or a difference of a power of 2 and a power of 3. Pema tells Yeshi that even though he has written the first fifteen prime numbers as the sum or the difference of a power of 2 and a power of 3, his argument does not constitute a proof because it does not prove the statement for *all* prime numbers.

To prove her point, Pema asks Yeshi to write 53 as the sum or the difference of a power of 2 and a power of 3. However, he is unable to, even after trying to for ten minutes. Because of this, Pema thinks that 53 cannot be expressed as the sum or the difference of a power of 2 and a power of 3. Your goal is to help Pema prove that this task is impossible.

(b) Find, with proof, all possible values of a power of 2 modulo 120. Then, find all possible values of a power of 3 modulo 120.

(c) Use your result from part (b) to determine all possible values of $2^a - 3^b$ modulo 120, where $a$ and $b$ are nonnegative integers.

(d) Prove that 53 is not the sum of a power of 2 and a power of 3. Then, use your result from part (c) to prove that 53 cannot be the difference of a power of 2 and a power of 3 (in some order).

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*Solution.*

(a) The next ten prime numbers can be written as the sum or the difference of a power of 2 and a power of 3 in the following ways:[1]

$$\begin{aligned}
13 &= 2^2 + 3^2 = 2^4 - 3^1 = 2^8 - 3^5 \\
17 &= 2^3 + 3^2 = 2^4 + 3^0 = 3^4 - 2^6 \\
19 &= 2^4 + 3^1 = 3^3 - 2^3 \\
23 &= 2^5 - 3^2 \\
29 &= 2^1 + 3^3 = 2^5 - 3^1 \\
31 &= 2^2 + 3^3 = 2^5 - 3^0 \\
37 &= 2^6 - 3^3 \\
41 &= 2^5 + 3^2 \\
43 &= 2^4 + 3^3 \\
47 &= 2^7 - 3^4.
\end{aligned}$$

(b) The possible values of a power of 2 modulo 120 are 1, 2, 4, 8, 16, 32, and 64. These are the only possible values because $2 \cdot 64 \equiv 8 \pmod{120}$, which is already a power of 2.

Similarly, the possible values of a power of 3 modulo 120 are 1, 3, 9, 27, and 81; these are the only possible values because $3 \cdot 81 \equiv 3 \pmod{120}$, which is already a power of 3.

(c) The possible values of $2^a - 3^b$ modulo 120 can be found by subtracting every possible value for a power of 3 modulo 120 from every possible value for a power of 2 modulo 120. Since there are 7 possible values for a power of 2 modulo 120 and 5 possible values for a power of 3 modulo 120, there are $7 \times 5 = 35$ possibilities. These possibilities are given in the table below, where each entry is the difference of its row and its column.

[1] For each prime number, only one representation is necessary for a complete solution.

| $-$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 3 | 7 | 15 | 31 | 63 |
| 3 | 118 | 119 | 1 | 5 | 13 | 29 | 61 |
| 9 | 112 | 113 | 115 | 119 | 7 | 23 | 55 |
| 27 | 94 | 95 | 97 | 101 | 109 | 5 | 37 |
| 81 | 40 | 41 | 43 | 47 | 55 | 71 | 103 |

Thus, the possible values are 0, 1, 3, 5, 7, 13, 15, 23, 29, 31, 37, 40, 41, 43, 47, 55, 61, 63, 71, 94, 95, 97, 101, 103, 109, 112, 113, 115, 118, and 119 modulo 120.

(d) 53 is not the sum of a power of 2 and a power of 3 because none of $53 - 3^0 = 52$, $53 - 3^1 = 50$, $53 - 3^2 = 44$, and $53 - 3^3 = 26$ are powers of 2.

To prove that 53 is not the difference of a power of 2 and a power of 3, first observe that neither 53 nor $-53 = 67 \pmod{120}$ is one of the possible values from part (c). Therefore, it is impossible for $2^a - 3^b$ or $3^b - 2^a$ to equal 53 modulo 120, so it is impossible for it to equal 53.