Final Exam

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Name: ____________________________

Date: _____________________________

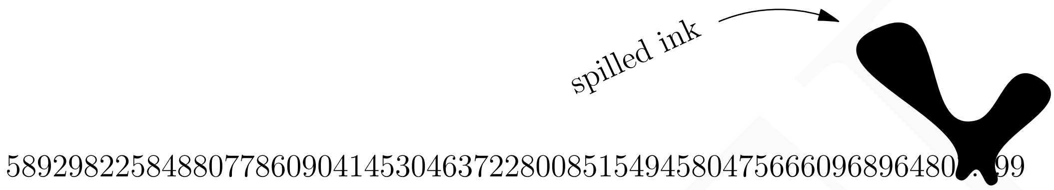
This final is an exam designed to measure how well you are able to tackle problems that use ideas that you likely have not seen before. Each problem is easier than the difficulty of the easiest IMO problems, but harder than the hardest problems covered at this camp so far. There are four problems: one for each IMO subject. Due to the difficulty of the problems, each problem has four parts which serve as hints to complete the problem.

Each part of each question is worth 7 points, so the exam is worth a total of $7 \times 4 \times 4 = 112$ total points. The questions are not sorted in order of difficulty, and for some questions it is possible to complete later parts without completing the previous parts. Because of this, be sure to attempt all parts of all questions.

You will have **4 hours and 30 minutes** for this exam. Please write up your solutions clearly and concisely. **All questions require a proof for full credit**, even if the problem statement does not say this explicitly. Even for problems that you cannot solve, please write down any partial progress or ideas that you may have. Enjoy the problems!

DO NOT OPEN THIS TEST BOOKLET UNTIL YOUR PROCTOR INSTRUCTS YOU TO

Problem 1. Jigme is writing the decimal expansion of $(2+\sqrt{3})^{108}$ with a pen. Unfortunately, her pen breaks in half after she writes the first 65 digits. The pen ink spills on her paper, covering the ones digit and the tenths digit of the number. Her paper looks like this:



The goal of this problem is to help Jigme determine the covered digits.

(a) Prove that

$$x^{n+1}+y^{n+1}=(x+y)(x^n+y^n)-(xy)(x^{n-1}+y^{n-1})$$

for every positive integer n. (Hint: don't use induction.)

(b) Let $s_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$. Using the identity from part (a), prove that $s_1 = 4$, $s_2 = 14$, and $s_{n+1} = 4s_n - s_{n-1}$ for all integers $n \geq 2$. (Hint: don't use induction.)

(c) Use the recurrence from part (b) to prove that s_{108} is an integer. Then, prove that $2 - \sqrt{3}$ is between 0 and 0.5. Finally, use both facts to determine the tenths digit of $(2 + \sqrt{3})^{108}$.

(d) Use the recurrence from part (b) to determine the sequence $s_1, s_2, \ldots$ modulo 10. Then, use your result to find the units digit of $(2+\sqrt{3})^{108}$.

Problem 2. Tandin, Tashi, Tenzin, Thinley, Tshering, and Tshewang are at a party, and each pair of people are either friends or strangers.

Call a group of people *mutual friends* if every pair of them are friends, and call a group of people *mutual strangers* if every pair of them are strangers. The goal of parts (a) and (b) is to prove that there is a group of three people at the party that are either mutual friends or mutual strangers.

(a) Prove that one of the two statements below must be true:

- There are three people at the party that are each friends with Tandin.
- There are three people at the party that are each strangers with Tandin.

(b) Consider the three people from part (a), along with Tandin. Prove that among these four people, three of them must be mutual friends or mutual strangers.

In fact, there must be *two* groups of three people that are mutual friends or mutual strangers, regardless of who are friends and who are strangers.[1] The goal of parts (c) and (d) is to prove this statement.

(c) Explain why proving this statement in the case that Tandin, Tashi, and Tenzin are mutual strangers implies the statement is true in all possible cases.

[1]It is possible for someone to be in both groups, but the groups cannot have the exact same set of people.

(d) Assuming that Tandin, Tashi, and Tenzin are mutual strangers, prove that some other group of three people are mutual friends or mutual strangers in the case that:

- no two people among Thinley, Tshering, and Tshewang are friends.
- some two people among Thinley, Tshering, and Tshewang are friends.

Problem 3. Gyeltshen encounters the following geometry problem on the Internet.

> Let ABC be an acute triangle with circumcenter O, and let point D lie on side $\overline{BC}$. Point E lies on $\overline{AC}$ such that $\overline{DE} \perp \overline{CO}$, and point F lies on $\overline{AB}$ such that $\overline{DF} \perp \overline{BO}$. Let K be the circumcenter of triangle AEF. Prove that $\overline{DK}$ and $\overline{BC}$ are perpendicular.

Gyeltshen starts solving the problem by setting $\angle BAC = \alpha$, but he gets stuck because his triangle looks equilateral and his diagram is both small and inaccurate.

(a) Help Gyeltshen draw a large and accurate diagram. Make sure that triangle ABC is acute and scalene. Then, express both $\angle BOC$ and $\angle EKF$ in terms of α.

(b) Gyeltshen, with the help of your large and accurate diagram, conjectures that $\angle BDF = \alpha$ and $\angle CDE = \alpha$. Using the result from part (a), help Gyeltshen prove this conjecture.

(c) Gyeltshen also notices that D, E, F, and K appear to be concyclic. Using the result from part (b), help Gyeltshen prove that D, E, F, and K are concyclic.

(d) Gyeltshen, using the result from part (c), draws a circle through points D, E, F, and K. Using this cyclic quadrilateral, prove that $\overline{DK}$ is perpendicular to $\overline{BC}$.

Problem 4. Yeshi thinks that every prime number can be expressed as the sum or the difference of a power of 2 and a power of 3, in some order. Pema thinks that Yeshi is wrong. The goal of this problem is to determine who is right.

(a) Yeshi is able to write the first five prime numbers as a sum or a difference of a power of 2 and a power of 3. However, Pema is not convinced. Help Yeshi write the next ten prime numbers as the sum or the difference of a power of 2 and a power of 3.

$$2 = 2^0 + 3^0$$

$$3 = 2^1 + 3^0$$

$$5 = 2^1 + 3^1$$

$$7 = 2^3 - 3^0$$

$$11 = 3^3 - 2^4$$

$$13 =$$

$$17 =$$

$$19 =$$

$$23 =$$

$$29 =$$

$$31 =$$

$$37 =$$

$$41 =$$

$$43 =$$

$$47 =$$

Pema is still not convinced that every prime number can be expressed as a sum or a difference of a power of 2 and a power of 3. Pema tells Yeshi that even though he has written the first fifteen prime numbers as the sum or the difference of a power of 2 and a power of 3, his argument does not constitute a proof because it does not prove the statement for *all* prime numbers.

To prove her point, Pema asks Yeshi to write 53 as the sum or the difference of a power of 2 and a power of 3. However, he is unable to, even after trying to for ten minutes. Because of this, Pema thinks that 53 cannot be expressed as the sum or the difference of a power of 2 and a power of 3. Your goal is to help Pema prove that this task is impossible.

(b) Find, with proof, all possible values of a power of 2 modulo 120. Then, find all possible values of a power of 3 modulo 120.

(c) Use your result from part (b) to determine all possible values of $2^a - 3^b$ modulo 120, where a and b are nonnegative integers.

(d) Prove that 53 is not the sum of a power of 2 and a power of 3. Then, use your result from part (c) to prove that 53 cannot be the difference of a power of 2 and a power of 3 (in some order).