Introduction to Counting

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Many mathematics problems, known as *counting* problems, involve counting the number of objects with a certain property. There are many techniques that can be used to count the number of such objects without listing them all out.

Problem 1. List the following:

(a) All two-digit numbers whose digits are both even

(b) All ways to award first place and second place in a competition between Karma, Pema, Sangay, Tashi, and Yeshi

(c) All ways to award a two-way tie for first place in a competition between Karma, Pema, Sangay, Tashi, and Yeshi

(d) All ways to arrange the letters in “PARO”

(e) All ways to arrange the letters in “GASA”

(f) All ways to arrange the letters in “MOMMY”

(g) All ways to arrange the letters in ”MAAMA”

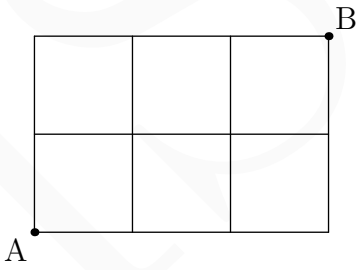
(h) All ways to arrange the letters in “KARMA” such that the two A’s are next to each other

(i) All ways to answer a four-question true-false test

(j) All three digit numbers with increasing digits ending in 6

(k) All ways to make fried rice, if the non-rice ingredients you have are garlic, chili, cilantro, and onions

(l) All ways to travel from point A to point B using five roads in the following diagram:



Problem 2.

(a) Yeshey has 3 shirts and 5 hats. How many outfits (consisting of a shirt and a hat) can he make from these?

(b) How many three-digit numbers have all even digits?

(c) How many three-digit numbers have distinct digits?

(d) How many ways are there to award first place, second place, and third place in a competition with ten contestants?

(e) A locker combination is a sequence of three distinct numbers between 1 and 50, inclusive. How many possible locker combinations are there?

(f) How many ways are there to arrange five people in a row?

Problem 3.

(a) How many ways are there to arrange the letters in "BHUTAN"?

(b) How many ways are there to arrange the letters in "THIMPHU"?

(c) How many ways are there to arrange the letters in "THIMPHU" such that the two H's are next to each other?

(d) How many ways are there to arrange the letters in "HIMALAYA"?

(e) How many ways are there to arrange the letters in "INDIAN"?

(f) How many ways are there to arrange the letters in "INDIAN" such that the two I's are next to each other?

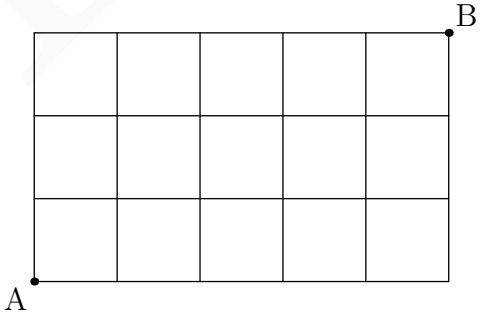
Problem 4.

(a) How many ways are there to award a three-way tie for first place in a competition with ten people?

(b) How many ways are there to select a group of four people from a class of ten people?

(c) How many three-digit numbers have their digits in increasing order?

(d) How many ways are there to travel from point A to point B using eight roads in the following diagram?



Problem 5.

(a) How many ways are there to answer a ten-question true-false test (without leaving any questions blank)?

(b) How many ways are there to answer a four-question multiple choice test (without leaving any questions blank), where each question has five answer choices?

(c) How many ways are there to make fried rice, if there are nine different non-rice ingredients available?

(d) How many ways are there to distribute 6 different gifts to 4 children?

(e) How many four-digit numbers do not contain the digit 0?

Some problems that ask for the number of objects with a certain property can be solved by counting the total number of objects and then subtracting the number of objects that don't satisfy the property. This method is known as *complementary counting*.

Problem 6.

(a) How many two-digit numbers are not a square number?

(b) How many two-digit numbers are not multiples of 5?

Problem 7.

(a) How many four-digit numbers contain the digit 0?

(b) How many four-digit numbers have at least one digit that is a 2 or a 3?

(c) How many four-digit numbers contain an even digit?

(d) How many four-digit numbers contain at least two digits that are the same?

Problem 8.

(a) How many ways are there to arrange the letters in "THIMPHU" such that the two H's are not next to each other?

(b) How many ways are there to arrange the letters in "INDIAN" such that the two I's are not next to each other?

For some counting problems, it is often useful to split the problem up into cases, solve the problem for each case, and sum the results to get the final answer. This method is known as *casework*.

Problem 9.

(a) How many positive integers less than 10000 have all even digits?

(b) How many positive integers less than 10000 do not contain the digit zero?

(c) How many positive integers less than 10000 have all distinct digits?

(d) How many positive integers less than 10000 have their digits in increasing order?

Problem 10.

(a) How many ways are there to make Nu 16 with Nu 10, Nu 5, and Nu 1 bills?

(b) How many ways are there to make Nu 20 with Nu 10, Nu 5, and Nu 1 bills?

(c) In American football, points can be scored in three ways. A *touchdown* is 7 points, a *field goal* is 3 points, and a *safety*: 2 points. A team finished a game with 24 points. How many combinations of scoring methods could lead to this score? The order of the scoring methods does not matter.

Problem 11.

(a) How many ways are there to write 6 as an ordered sum of odd numbers?

(b) How many ways are there to write 7 as an ordered sum of odd numbers?

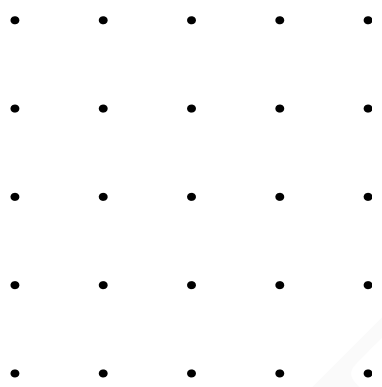
There is no single recipe to solve all counting problems. Many complex counting problems require a combination of different methods, including (but not limited to) complementary counting and casework that we discussed above. In addition, there are often multiple methods to solve a single problem. In the following problems, choose the method wisely.

Problem 12.

(a) How many ways are there to choose three distinct digits such that at least one digit is odd?

(b) How many ways are there to choose four numbers from the list $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ such that at least one prime number is chosen?

Problem 13. Consider the dots of grid shown below.



(a) Count the number of squares formed by any four dots whose sides are parallel to the grid line.

(b) Count the number of rectangles formed by any four dots whose sides are parallel to the grid line.

(c) Count the number of squares formed by any four dots.