Area

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Area, the amount of space a two-dimensional object takes up, is a fundamental notion in geometry. This handout contains problems that ask for the area of various shapes, as well as problems whose solutions involve area in some way.

Problem 1.

(a) Prove that the area of a triangle with base b and height h is $\frac{1}{2}bh$.

(b) Prove the area of a parallelogram with base b and height h is bh.

(c) Prove the area of a trapezoid with bases a, b and height h is $\frac{a+b}{2}h$.

(d) Prove that the area of an equilateral triangle with side s is $\frac{\sqrt{3}}{4}s^2$.

(e) Prove that the area of a regular hexagon with side s is $\frac{3\sqrt{3}}{2}s^2$.

(f) Prove that the area of a circle is πr^2, where π is the ratio of its circumference to its diameter.

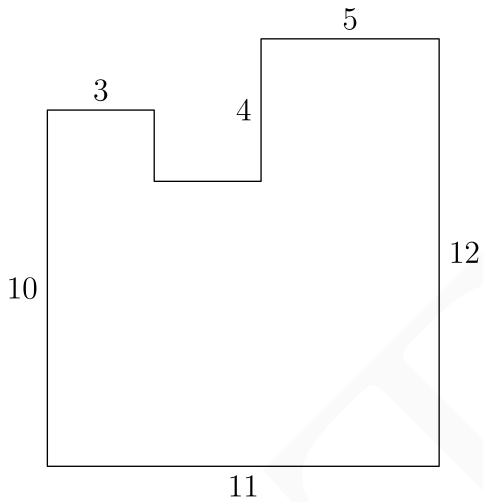
(g) Prove that the area of a triangle with sides a, b, and c is rs, where $s = \frac{1}{2}(a+b+c)$ and r is its inradius.

(h) Prove that the area of a triangle with sides a, b, and c is

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$.

Problem 2. Find the perimeter and the area of the following octagon, whose angles are all right angles.



Problem 3.

(a) The diagonal of a square has length 6. Find its area.

(b) The shortest diagonal of a regular hexagon has length 6. Find its area.

Problem 4. An isosceles right triangle $\mathcal{T}$, a square $\mathcal{S}$, and a circle $\mathcal{C}$ each have area 1. Sort them by perimeter, from smallest to largest.

Problem 5. Let $\mathcal{T}$ be a triangle with sides 13, 14, and 15.

(a) Find its area.

(b) Find the lengths of the three altitudes of $\mathcal{T}$.

(c) Find its inradius.

Problem 6. Let $\mathcal{T}$ be a triangle with sides 13, 20, and 21.

(a) Find its area.

(b) Find the lengths of the three altitudes of $\mathcal{T}$.

(c) Find its inradius.

Problem 7. Let $ABCDEF$ be a regular hexagon with area 6.

(a) What is the area of triangle ABC?

(b) What is the area of triangle ABD?

(c) What is the area of triangle ACE?

Problem 8.

(a) What is the area of the set of all points within 1 unit of a line segment of length 4?

(b) What is the area of the set of all points within 1 unit of the boundary of a 4 by 4 square?

(c) What is the area of the set of all points within 1 unit of the interior of an equilateral triangle with side length 4?

When two triangles share a common height, the ratio of their areas equals the ratio of their base lengths. This fact can be used to solve many area problems.

Problem 9. In the following problem, let $[\mathcal{X}]$ denote the area of $\mathcal{X}$.

(a) Let X, Y, and Z lie on a line in this order, and let P be any point not on this line. Prove that

$$\frac{[PXY]}{[PYZ]} = \frac{XY}{YZ}.$$

(b) Let $ABCD$ be a quadrilateral whose diagonals meet at X. Prove that

$$\frac{[ABC]}{[ADC]} = \frac{BX}{DX}.$$

Problem 10.

(a) Let ABC be a triangle with area 60. Let X be a point on segment $\overline{BC}$ such that $BX : XC = 1 : 2$. Find the area of triangle ABX.

(b) Let ABC be a triangle with area 60, and let M be the midpoint of $\overline{BC}$. Let P be a point on segment $\overline{AM}$ such that $AP : PM = 4 : 1$. Find the area of triangle APB.

(c) Let ABC be a triangle with area 60. Let X be the point on $\overline{AC}$ such that $AX : XC = 3 : 2$. Let Y be the point on segment $\overline{AB}$ such that $AY : YB = 4 : 5$. Find the area of triangle AXY.

(d) Let $ABCD$ be a parallelogram with area 60. Let X be the point on AB such that $AX : XB = 1 : 2$. Find the area of triangle XBC.

(e) Let $ABCD$ be a parallelogram with area 60. Let X be the point on AB such that $AX : XB = 1 : 2$. Let M be the midpoint of $\overline{CX}$. Find the area of triangle ADM.

(f) Let $ABCD$ be a rectangle with $AB = 20$ and $AD = 24$. Let M be the midpoint of $\overline{CD}$, and let X be the reflection of M across point A. Find the area of triangle XBD.

> Whenever there is a triangle, shifting one vertex along a line parallel to the opposite side gives a triangle with equal area.

Problem 11. Let $ABCD$ be a trapezoid such that $\overline{AB} \parallel \overline{CD}$.

(a) Prove that triangles ACD and BCD have the same area.

(b) Suppose that $\overline{AC}$ and $\overline{BD}$ intersect at X. Prove that triangles XAD and XBC have the same area.

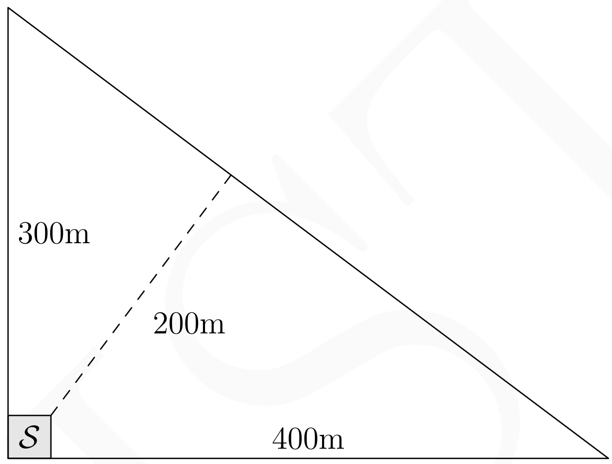
Problem 12.

(a) Let $ABCDE$ be a convex pentagon such that $\overline{AB} \parallel \overline{CE}$ and $\overline{AD} \parallel \overline{BC}$. Prove that triangles ABE and BCD have the same area.

(b) Let $ABCD$ be a square with side length 10. Point E is on ray $\overline{AB}$ such that $AE = 14$, and point F is on ray $\overline{AD}$ such that $AF = 17$. The line through B parallel to $\overline{CE}$ and the line through D parallel to $\overline{CF}$ meet at P. Find the area of quadrilateral $AEPF$.

> Sometimes, unexpected applications of area can appear in a problem although the problem does not mention areas.

Problem 13. Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 300 meters and 400 meters. In the corner where those sides meet at a right angle, he leaves a small unplanted square $\mathcal{S}$ so that it looks like a right angle symbol.



(a) Find the distance from the bottom-left corner of $\mathcal{S}$ to the hypotenuse.

(b) The distance from the top-right corner of $\mathcal{S}$ to the hypotenuse is 200 meters. What is the side length of $\mathcal{S}$?

Problem 14 (Vivani's Theorem). Let $\mathcal{T}$ be an equilateral triangle and let P be a point inside it. Prove that the sum of the distances from P to the three sides of $\mathcal{T}$ equals the length of an altitude of $\mathcal{T}$.

Problem 15 (Angle Bisector Theorem). Let ABC be a triangle.

(a) Suppose that the internal angle bisector of $\angle BAC$ meets $\overline{BC}$ at D. Prove that $AB : AC = BD : DC$.

(b) Suppose that the external angle bisector of $\angle BAC$ meets $\overline{BC}$ at E. Prove that $AB : AC = BE : EC$.

Problem 16 (Ceva's Theorem[1]). Let ABC be a triangle. Points D, E, and F lie on $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively such that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent. Prove that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

[1] In fact, it is also true that if $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$, then $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent. This has many applications, one of which is to prove that the three mediare concurrent.