Proofs in Angle Chasing

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This is a continuation of the Introduction to Angle Chasing handout. This handout is focused on proving geometrical statements via angle chasing. Many problems in this handout are adapted from famous lemmas and results that have appeared in mathematical competitions.

Olympiad geometry problems will consist of just statements. The first thing a student must do is to draw a diagram, preferably with a compass or a ruler. When drawing the diagram, do not make any assumptions about the problem. Instead, try to make any false assumption look as false as possible.

Problem 1. Draw the following diagrams, following the general rules given above:

(a) Let ABC be a triangle. Let D be the foot of the altitude from A to $\overline{BC}$.

(b) Let ABC be an equilateral triangle. Let D be the foot of the altitude from A to $\overline{BC}$.

(c) Let ABC be an triangle satisfying $AB = AC$. Let D be the foot of the altitude from A to $\overline{BC}$.

(d) Let ABC be an isosceles triangle such that $AB = BC$. Let D be the foot of the altitude from A to $\overline{BC}$.

(e) Let ABC be an equilateral triangle and $BCDE$ be a square such that ABC lies inside $BCDE$.

(f) Let $ABCD$ be a rectangle inscribed in a circle Ω. The tangent to Ω at B meets $\overline{AC}$ at X.

(g) Let ABC be an acute triangle. Let O be the circumcenter. Let D be the foot of the altitude from A to $\overline{BC}$. Let the angle bisector of $\angle BAC$ meet BC at E. Let M be the midpoint of $\overline{BC}$.

Recall the following facts:

- P is equidistant to A and B if and only if P lies on the perpendicular bisector of AB.
- P is equidistant to lines ℓ and m if and only if P lies on an angle bisector of ℓ and m.

Problem 2. For an acute triangle ABC, draw the following in a single diagram:

- the *circumcenter* O, which is equidistant from all three vertices;
- the *incenter* I, which is the intersection of three angle bisectors;
- the *orthocenter* H, which is the intersection of three altitudes; and
- the *centroid* G, which is the intersection of three medians.[1]

[1] It is not obvious that each triplet of lines concur. The concurrency of the altitudes will be proven later in this handout.

Recall the following results:

- Let ABC be a triangle and let O be the circumcenter of triangle ABC. Then $\angle BOC = 2\angle BAC$.
- Let Ω be a circle with diameter $\overline{PQ}$, and let X be a point on Ω. Then $\angle PXQ = 90^\circ$.
- Let $ABCD$ be a cyclic quadrilateral. Then $\angle BAC = \angle BDC$ and $\angle A + \angle C = 180^\circ$.
- Let Ω be a circle with center O, and let line AT be tangent to Ω at A. Then $\angle OAT = 90^\circ$.
- Let ABC be a triangle. Let T be a point such that T and C are on the same side of $\overline{AB}$, and $\overline{AT}$ is tangent to the circumcircle of ABC. Then $\angle TAC = \angle ABC$.

Problem 3. Let ABC be a triangle such that $\angle BAC = 90^\circ$, and let $\overline{AD}$ be an altitude of triangle ABC. Prove that $\angle BAD = \angle ACB$.

Problem 4. Let ABC be an acute triangle with circumcenter O, and let $\overline{AD}$ be an altitude of triangle ABC. Prove that $\angle BAD = \angle OAC$.

Problem 5. Let O be the circumcenter of an acute triangle ABC such that $AC > BC$. Let the circumcircle of triangle BOC intersect $\overline{AC}$ again at X. Prove that $AX = XB$.

Problem 6. The incircle of triangle ABC touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F. Prove that $\angle EDF = 90^\circ - \frac{1}{2}\angle BAC$.

Problem 7. Let ABC be an acute triangle such that $AB < AC$. The angle bisector of $\angle BAC$ meets $\overline{BC}$ at D. The tangent at A to the circumcircle of triangle ABC meets $\overline{BC}$ at T. Let $\angle ABC = b$ and let $\angle ACB = c$.

(a) Express $\angle ATB$ in terms of b and c.

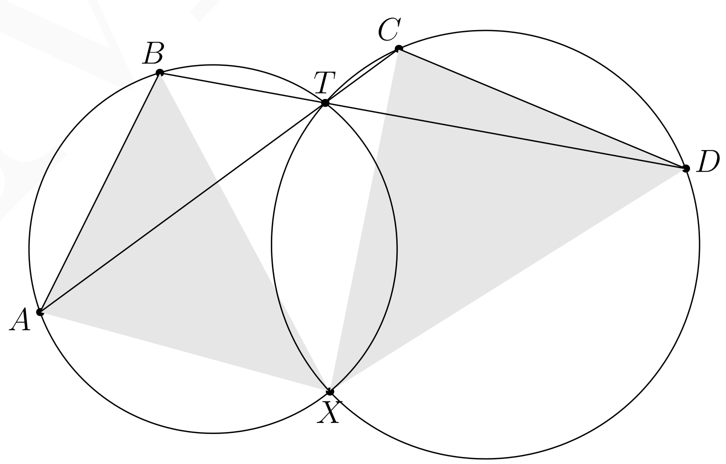
(b) Express $\angle ADB$ in terms of b and c.

(c) Prove that $TA = TD$.

Recall that triangle ABC is similar to XYZ if and only if $\angle A = \angle X$ and $\angle B = \angle Y$.

Problem 8. Let ABC be a triangle. Point P lies on side $\overline{BC}$, and point Q lies on arc BC (not containing A) of the circumcircle of triangle ABC. Given that $\angle BAP = \angle CAQ$, prove that triangles ABP and ACQ are similar.

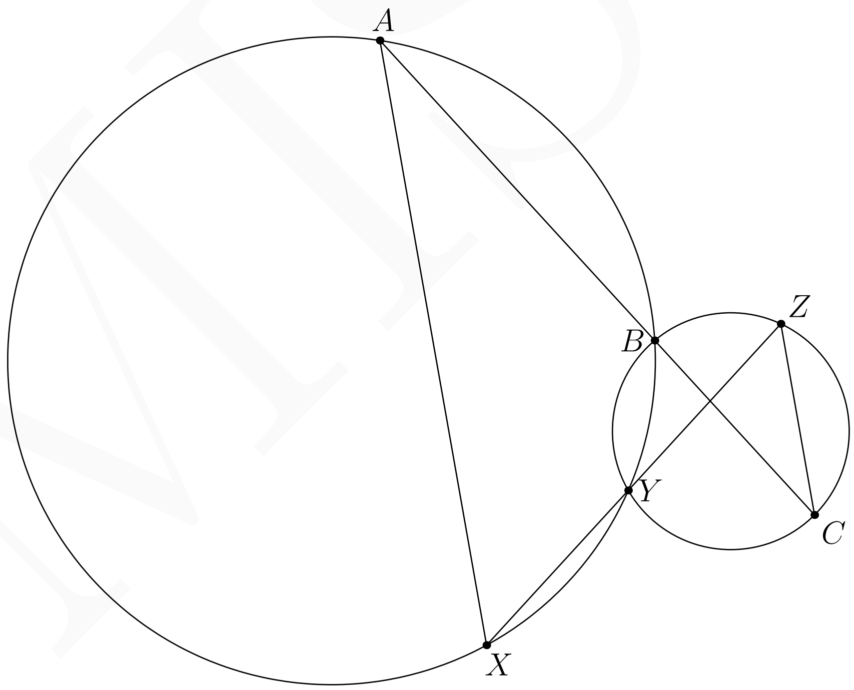
Problem 9 (Spiral Similarity Lemma). Let $\overline{AB}$ and $\overline{CD}$ be two segments, and let $\overline{AC}$ and $\overline{BD}$ meet at T. Let the circumcircles of triangles ABT and CDT meet again at X. Prove that triangles XAB and XCD are similar.



The following are variants of facts we have seen so far but adapted for proving statements in geometry.

- Let $ABCD$ be a quadrilateral. If $\angle A + \angle C = 180°$, then $ABCD$ is cyclic.
- Let $ABCD$ be a quadrilateral. If $\angle BCA = \angle BDA$, then $ABCD$ is cyclic.
- Let ABC be a triangle and let T be a point such that T and C are on the same side of $\overline{AB}$. If $\angle TAC = \angle ABC$, then AT is tangent to the circumcircle of triangle ABC.

Problem 10 (Reim's Theorem). In the diagram below, quadrilateral $ABXY$ is cyclic, quadrilateral $BCYZ$ is cyclic, B lies on $\overline{AC}$, and Y lies on $\overline{XZ}$. Prove that $\overline{AX} \parallel \overline{CZ}$.



Problem 11 (Miquel's Theorem). Let D, E, and F be points on sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ of triangle ABC. Suppose that the circumcircles of triangles BDF and CDE meet again at P. Prove that P lies on the circumcircle of AEF.[2]

Let ABC be an acute triangle with incenter I and orthocenter H. Recall that:

- $\angle BIC = 90^\circ + \frac{1}{2}\angle A$.
- $\angle BHC = 180^\circ - \angle BAC$.

Problem 12 (Incenter-Excenter Lemma). Let ABC be a triangle with incenter I. Let M be the circumcenter of triangle BIC.

(a) Prove that quadrilateral $ABMC$ is cyclic.

(b) Prove that $\overline{AM}$ bisects $\angle BAC$.

[2]This is equivalent to stating that the circumcircles of triangles AEF, BDF, and CDE share a common point.

Problem 13 (Orthocenter Reflections). Let ABC be a triangle with orthocenter H.

(a) Prove that the reflection of H over $\overline{BC}$ lies on the circumcircle of triangle ABC.

(b) Prove that the reflection of H over the midpoint of $\overline{BC}$ lies on the circumcircle of triangle ABC.

Recognizing cyclic quadrilaterals is one of the most fundamental skills in olympiad Geometry. Cyclic quadrilaterals allow problem solvers to relate different angles in the problem.

Problem 14.

(a) Let $ABCD$ be a quadrilateral such that $\angle A = \angle C = 90^\circ$. Explain why $ABCD$ is cyclic.

(b) Let P be a point inside equilateral triangle ABC such that $\angle PBC = 35^\circ$ and $\angle PCB = 20^\circ$. Let D, E, and F be the feet from P to $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Determine $\angle EDF$.

(c) Let $ABCD$ be a square. Let M be the midpoint of $\overline{BC}$. Let P be the point on side $\overline{CD}$ such that $\overline{AM} \perp \overline{PM}$. If $\angle AMB = \theta$, express $\angle AED$ in terms of θ.

(d) Let $ABCDE$ be a convex pentagon such that $\angle A = 90^\circ$ and $BCDE$ is a square with center O. Prove that $\overline{AO}$ bisects $\angle BAE$.

Problem 15 (Simson Line). Let P lie on the circumcircle of triangle ABC. Prove that the three feet from P to the three sides of ABC are collinear.

Problem 16 (Orthocenter and Orthic Triangle). Let ABC be an acute triangle, and let $\overline{BE}$ and $\overline{CF}$ be altitudes of triangle ABC. Let H be the intersection of $\overline{BE}$ and $\overline{CF}$, and let D be the intersection of $\overline{AH}$ and $\overline{BC}$.

(a) Prove that $\overline{AH} \perp \overline{BC}$. Thus, we have shown that the three altitudes are concurrent.

(b) List all cyclic quadrilaterals formed by four points in the diagram.

(c) Prove that $\overline{DH}$ bisects $\angle EDF$.

Problem 17 (Right Angle in Intouch Chord Lemma). Let I be the incenter of triangle ABC. The incircle of triangle ABC touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively. Let X be the foot from C to $\overline{BI}$.

(a) List all cyclic quadrilaterals in the diagram. Did you notice a cyclic pentagon?

(b) Prove that X lies on $\overline{EF}$.