

## Introduction to Taylor Series

Why are we looking at power series? If we reverse the equation for the geometric series:

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

we get a description of  $\frac{1}{1-x}$  in terms of a series. In fact, we can represent all of the functions we've encountered in this course in terms of series.

The technique is similar to the use of a decimal expansion to represent  $1/3$  or  $\sqrt{2}$ . When we describe a function like  $e^x$  or  $\arctan x$  in terms of a series we can approximate and manipulate those functions as easily as we do polynomial functions.

### Rules for convergent power series

What sorts of manipulations might we want to perform? Addition, multiplication, division, substitution (composition), integration and differentiation. The rules for manipulation of power series are essentially the same as those for manipulating polynomials!

$$f(x) + g(x), \quad f(x) \cdot g(x), \quad f(g(x)), \quad f(x)/g(x), \quad \frac{d}{dx}f(x), \quad \int f(x)dx$$

We can do all of these with power series; in this class integration and differentiation will be the most interesting manipulations.

We take the derivative of a power series just as we do for polynomials:

$$\frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) = a_1 + 2a_2x + 3a_3x^2 + \dots$$

Similarly, the formula for the integral of a power series is:

$$\int (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)dx = c + a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + \frac{a_3x^4}{4} + \dots$$

Here the arbitrary constant  $c$  takes the place of the constant term in the new series.

**Question:** Is that a series or a polynomial?

**Answer:** It's a polynomial if it ends; if it goes on infinitely far then it's a series.

**Question:** You can add up terms of  $x$  in a series?

**Answer:** When we introduced series we described them as infinite sums of *numbers*. At the start of this class we rewrote the geometric series using the *variable*  $x$  in place of the "constant value"  $a$ . When we plug in a value for  $x$  we get a sum of infinitely many numbers, so as long as we remember that  $x$  is a placeholder for a numerical value there's no problem.

In other words, what we're working with here are functions of  $x$ . These functions are defined for values of  $x$  inside the radius of convergence and undefined

for values of  $x$  that are too large, just as the function  $f(x) = \sqrt{x}$  is defined for positive values of  $x$  and undefined for  $x < 0$ .

Power series are infinite sums of powers of  $x$ , with coefficients. People also study and use series that are infinite sums of sines and cosines and lots of other series, but we're only going to study power series here.