Exploring the Trapezoidal Rule

Compare the trapezoidal rule to the left Riemann sum. The area of each trapezoid is calculated using twice as much information as the area of each rectangle in the Riemann sum. In this sense, the trapezoidal rule is twice as good as the left Riemann sum.

We could also say that the left Riemann sum with $n = 8$ partitions is twice as good as the left Riemann sum with $n = 4$ partitions.

Use the mathlet at `http://math.mit.edu/~jmc/daimp/RiemannSums.html` to compare the result of applying the trapezoidal rule with $n = 4$ to the result of using the left Riemann sum with $n = 8$. Are the results equal? Why or why not?

Solution

Using the mathlet, you should have found that for $f(x) = x^2 - 2x$ the left Riemann sum with $n = 8$ was:

$$0.63281$$

while applying the trapezoidal rule with $n = 4$ yields:

$$0.28125.$$

Clearly these results are not equal.

In fact, we should not expect them to be equal. Figure 1 shows the result of subdividing the (carefully chosen) graph of $\cos(\pi x) + 1$ over the interval $[0, 8]$ into 4 regions. In this case both the trapezoidal rule and the left Riemann sum give an estimate of $2 \cdot 8 = 16$ for the area under the curve; both methods are equally bad.

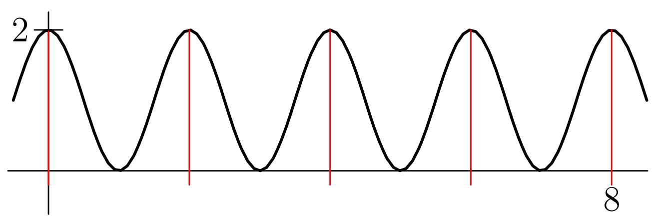

Figure 1: The graph of $y = \cos(\pi x) + 1$ over $[0, 8]$, divided into 4 regions.

If we instead subdivide the interval into 8 regions (Figure 2) we see that the left Riemann sum gives the much more accurate estimate of $2 \cdot 4 = 8$. It turns out that the trapezoidal rule also gives an area estimate of 8 in this case.

We started with the assumption that the trapezoidal rule with $n = 4$ would give twice as good an estimate as the left Riemann sum with $n = 4$, as would the left Riemann sum with $n = 8$. What's wrong with these assumptions?

Although computing the area of each individual trapezoid requires more information than computing the area of a rectangle, the trapezoidal rule as a

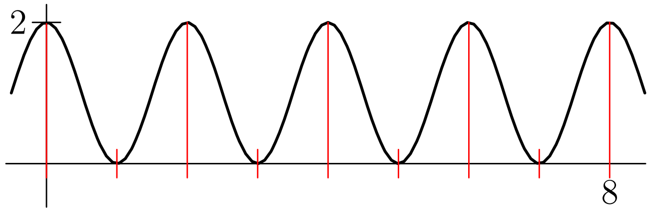

Figure 2: The graph of $y = \cos(\pi x) + 1$ over $[0, 8]$, divided into 8 regions.

whole uses only one more piece of information than the left Riemann sum does. Compare the two formulas:

$$\begin{aligned}\text{trapezoidal rule estimate} &= \Delta x \left(\frac{y_o}{2} + y_1 + y_2 + ... + y_{n-1} + \frac{y_n}{2} \right), \\ \text{left Riemann sum estimate} &= \Delta x \left(y_0 + y_1 + y_2 + ... + y_{n-1} \right).\end{aligned}$$

For a large enough number n of subdivisions, the results obtained using the trapezoidal rule are very similar to those obtained using Riemann sums; verify this using the mathlet. As Professor Jerison mentioned in lecture, the estimate given by the trapezoidal rule is exactly equal to the average of the left Riemann sum and the right Riemann sum.

In contrast, doubling the number of subdivisions does approximately double the amount of information used in an approximation and so significantly improves the estimate's likelihood of being close to the exact answer.

Finally, the notion that two estimates made with equal accuracy must be equal is incorrect. The values 0.28 and −0.28 are equally accurate estimates of $\int_{-1}^{2} (x^2 - 2x)\, dx = 0$, but the two estimates aren't equal to each other.