Quadratic Approximation

Last class we derived a list of quadratic approximations for values of x near 0:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \quad (x \approx 0)$$

- $\sin x \approx x \quad$ (if $x \approx 0$)
- $\cos x \approx 1 - \frac{x^2}{2} \quad$ (if $x \approx 0$)
- $e^x \approx 1 + x + \frac{1}{2}x^2 \quad$ (if $x \approx 0$)
- $\ln(1+x) \approx x - \frac{1}{2}x^2 \quad$ (if $x \approx 0$)
- $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 \quad$ (if $x \approx 0$)

We'll use these in two examples, then finish deriving the final two items on the list.

Here's an example of the power of linear approximation, and of what quadratic approximation can do for us that linear approximation cannot.

Recall that when we discussed exponential and logarithmic functions we said that:

$$a_k = \left(1 + \frac{1}{k}\right)^k$$

tends to e as k goes to infinity. We did that by taking

$$\ln a_k = k \ln\left(1 + \frac{1}{k}\right).$$

That was a fairly difficult calculation which is made much easier by linear approximation. Since the linear approximation of $\ln(1+x)$ is just x,

$$\ln a_k = k \ln\left(1 + \frac{1}{k}\right) \approx k(1/k) = 1.$$

Can we really do this? The linear appoximation only works when x is near 0, but as k goes to infinity $\frac{1}{k}$ is indeed near 0. So as k approaches infinity, the linear approximation gets closer and closer to the exact value of the function, and $\ln a_k$ approaches 1. This is an example of how we want to use approximation.

Now if we want to find the *rate* of convergence — if we want to find out how fast the value of $k \ln\left(1 + \frac{1}{k}\right)$ approaches 1 — we need to look at how big $\ln a_k - 1$ for large values of k. To do this you'll use quadratic approximation; the formula for the quadratic approximation of the natural log function is:

$$\ln(1+x) \approx x - \frac{1}{2}x^2 \quad \text{(for } x \text{ near 0).}$$

You need the next higher order term to get a more detailed understanding of $\lim_{k\to\infty} \ln a_k$; this question is on the homework.

Question: How do we know when to use a linear approximation and when to use a quadratic one?

Answer: That's a very good question. For now, when I give you a question, I'll specify whether I want you to use a linear or a quadratic approximation. As time goes on, I'd like you to get a feel for when you can get away with a linear approximation. You should only use a quadratic approximation if somebody forces you to; always start trying with a linear one. Because the quadratic ones are much more complicated as you'll see in this next example.

In real life when you're faced with a problem like this — maybe some satellite is orbiting and you want to know the effects of gravity or something like that — nobody is going to tell you anything. They're not even going to tell you whether a linear approximation is relevant; you're on your own.

Last lecture we computed the linear approximation for x near 0 of

$$\frac{e^{-3x}}{\sqrt{1+x}} = e^{-3x}(1+x)^{-1/2}.$$

This lecture we'll compute a quadratic approximation for this function when x is near 0.

To do this we need to use the quadratic approximations for e^{-3x} and $(1+x)^{-1/2}$. We'll use the following two approximation formulas:

$$\begin{aligned} e^x &\approx 1 + x + \frac{1}{2}x^2 \\ (1+x)^r &\approx 1 + rx + \frac{r(r-1)}{2}x^2 \end{aligned}$$

substituting $x = 3x$ into the first and $r = -\frac{1}{2}$ into the second.

$$e^{-3x}(1+x)^{-1/2} \approx \left(1 + (-3x) + \frac{1}{2}(-3x)^2\right)\left(1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2\right)$$

This looks awful! But we can ignore any terms of higher degree than x^2 and avoid doing those multiplications when we apply the distributive law, so it's not as bad as it looks.

$$e^{-3x}(1+x)^{-1/2} \approx 1 - 3x - \frac{1}{2}x + \frac{3}{2}x^2 + \frac{9}{2}x^2 + \frac{3}{8}x^2$$

Now we combine like terms:

$$e^{-3x}(1+x)^{-1/2} \approx 1 - \frac{7}{2}x + \frac{51}{8}x^2$$

Remember that this approximation is only valid for $x \approx 0$, and notice that the first two terms are exactly the linear approximation we got last time.

As you can see, calculations with quadratic approximations are much more involved than those with just linear approximations.

Question: Why do we get to drop all the higher order terms?

Answer: Because in the situation in which we're going to apply this, x is a very small number like $\frac{1}{100}$. That means that $x^2 \approx \frac{1}{10000}$ and $x^3 \approx \frac{1}{1000000}$. We don't need an exact answer so we can safely ignore anything as small as a millionth, which is what our x^3 terms represent.

Now that we've seen a couple of examples of quadratic approximation, we'll derive the last two formulas in our library. The general formula for a quadratic approximation is:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \quad (x \approx 0)$$

As usual, we chose the base point $x_0 = 0$. Shown below are the first and second derivatives of the functions we're interested in and their values at $x_0 = 0$. Combining this with the general formula yields the quadratic approximations listed above.

$f(x)$	$f'(x)$	$f''(x)$	$f(0)$	$f'(0)$	$f''(0)$
$\sin x$	$\cos x$	$-\sin x$	0	1	0
$\cos x$	$-\sin x$	$-\cos x$	1	0	-1
e^x	e^x	3^x	1	1	1
$\ln(1+x)$	$\frac{1}{1+x}$	$\frac{-1}{(1+x)^2}$	0	1	-1
$(1+x)^r$	$r(1+x)^{r-1}$	$r(r-1)(1+x)^{r-2}$	1	r	$r(r-1)$

In practice, we can approximate most of the functions we encounter using algebraic combinations of the functions in this library.