Cross product

The cross product is another operation you can apply to two vectors in three dimensional space — it is specific to three dimensions.

The *cross product* of two vectors **A** and **B** in three dimensional space is the vector:

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.\end{aligned}$$

Be careful to write your multiplication symbol carefully so that you don't get the cross product confused with the dot product; the cross product is a vector and the dot product is not. Notice that not all of the entries in this determinant are numbers — this disagrees with the rules we set up when defining the determinant. We're just using the symbolic notation of a determinant here to help us remember the actual formula, which appears on the next line.

As usual, we ask the question "What is it good for?"

Theorem: Geometric interpretation of the cross product.

- The length $|\mathbf{A}\times\mathbf{B}|$ equals the area of the parallelogram formed by **A** and **B**. This looks like a complicated formula, but it works! Note that we don't need to worry about the sign of the result; the length of a vector is always non-negative.
- The direction $\text{dir}(\mathbf{A}\times\mathbf{B})$ is perpendicular to the plane of that parallelogram in the direction given by the right hand rule, so is perpendicular to both **A** and **B**.

There are two directions parallel to the plane containing **A** and **B**. The *right hand rule* tells us which direction the cross product will point in. In France, the right hand rule is described in terms of a cork screw and bottle; here we'll use the more common version. To find the direction of $\mathbf{A}\times\mathbf{B}$, point your right hand in the direction of **A** then curl your fingers so that they point toward **B**. Stick your thumb straight out; the direction your thumb points is the direction of $\mathbf{A}\times\mathbf{B}$.

Here are some step by step instructions:

1. Right hand points parallel to **A**.
2. Fingers point parallel to **B**.
3. Thumb points parallel to $\mathbf{A}\times\mathbf{B}$.

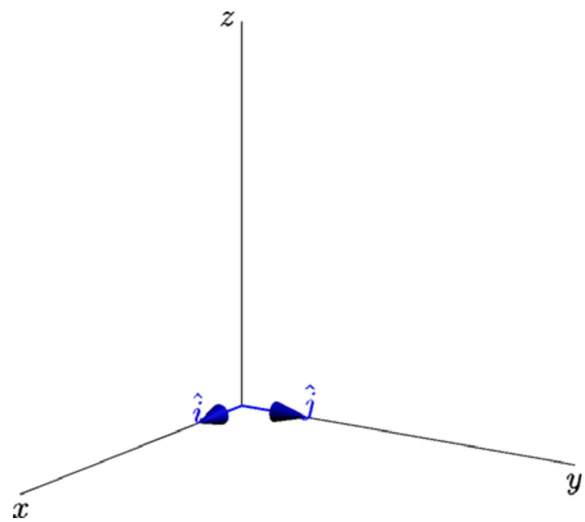

Figure 1: The unit vectors **i** and **j**.

The cross product $\mathbf{i} \times \mathbf{j}$ equals:

a) $\mathbf{k}$

b) $-\mathbf{k}$

c) 1

d) 0

e) I don't know

Make sure you use your right hand to answer this, not your left! Using the fact that **i** points toward us and **j** points to our right, we find that $\mathbf{i} \times \mathbf{j}$ (and our thumb) points upward, in the direction of **k**:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

The length of the cross product is 1 because that is the volume of a unit cube in three dimensional space.

Let's confirm this using our definition:

$$\begin{aligned}
\mathbf{i} \times \mathbf{j} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k}. \\
&= (0-0)\mathbf{i} - (0-0)\mathbf{j} + (1-0)\mathbf{k} \\
&= \mathbf{k}.
\end{aligned}$$

In this example it was easy to compute the cross product geometrically, but for more complicated vectors it's usually preferable to compute the cross product algebraically. Computing determinants is difficult at first, but it should become easier with practice.

Finding Volume Geometrically

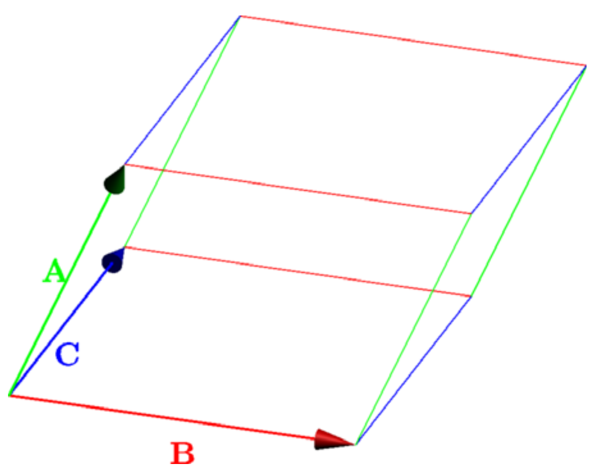

Figure 2: The parallelepiped described by **A**, **B** and **C**.

Let's return to the question of how to find volume without using a determinant. Recall that the area of a parallelepiped is the area of the base times the height. How can we use this formula in practice?

$$\text{Volume} = \text{area(base)} \cdot \text{height}$$

The base of the parallelepiped is the parallelogram with sides **B** and **C**. We just learned that the area of this parallelogram is the length of the cross product $\mathbf{B} \times \mathbf{C}$.

The height will be the (positive) component of $\mathbf{A}$ in the direction $\widehat{n}$ perpendicular to the plane containing $\mathbf{B}$ and $\mathbf{C}$. We can find this vector $\widehat{n}$ by using $\mathbf{B} \times \mathbf{C}$. The cross product $\mathbf{B} \times \mathbf{C}$ points in the same direction as $\widehat{n}$ but is probably not a unit vector. We'll have to divide by its length to get a unit vector.

Algebraically, the calculation becomes:

$$\begin{aligned}
\text{Volume} &= \text{area(base)} \cdot \text{height} \\
&= |\mathbf{B} \times \mathbf{C}|(\mathbf{A} \cdot \widehat{n}) \\
&= |\mathbf{B} \times \mathbf{C}|(\mathbf{A} \cdot \frac{\mathbf{B} \times \mathbf{C}}{|\mathbf{B} \times \mathbf{C}|}) \\
\text{Volume} &= \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})
\end{aligned}$$

Note that we have to be careful what order we do these operations in. If we compute $\mathbf{A} \cdot \mathbf{B}$ first, we will get a scalar. We can't take the cross product of a scalar with $\mathbf{C}$, so we must compute $\mathbf{B} \times \mathbf{C}$ first.

But this is just the volume of a parallelepiped, so geometrically:

$$\det(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}).$$

Does this make sense algebraically?

$$\begin{aligned}
\det(\mathbf{A}, \mathbf{B}, \mathbf{C}) &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \left(- \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \right) + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}
\end{aligned}$$

Yes, the algebraic formulas agree that the volume of the parallelepiped is the same no matter how we compute it.

Soon we'll study matrices and equations of planes, but this is a good time to take a break and work some problems.