Equations of Planes

We have touched on equations of planes previously. Here we will fill in some of the details.

Planes in point-normal form

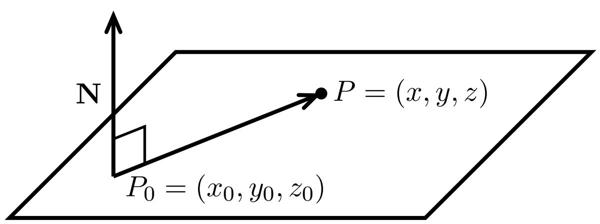
The basic data which determines a plane is a point P_0 in the plane and a vector $\mathbf{N}$ orthogonal to the plane. We call $\mathbf{N}$ a *normal* to the plane and we will sometimes say $\mathbf{N}$ is *normal* to the plane, instead of orthogonal.

Now, suppose we want the equation of a plane and we have a point $P_0 = (x_0, y_0, z_0)$ in the plane and a vector $\overrightarrow{\mathbf{N}} = \langle a, b, c\rangle$ normal to the plane.

Let $P = (x, y, z)$ be an arbitrary point in the plane. Then the vector $\overrightarrow{\mathbf{P_0P}}$ is in the plane and therefore orthogonal to $\mathbf{N}$. This means

$$\begin{aligned} & \mathbf{N} \cdot \overrightarrow{\mathbf{P_0P}} = 0 \\ \Leftrightarrow \quad & \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0 \\ \Leftrightarrow \quad & a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \end{aligned}$$

We call this last equation the point-normal form for the plane.



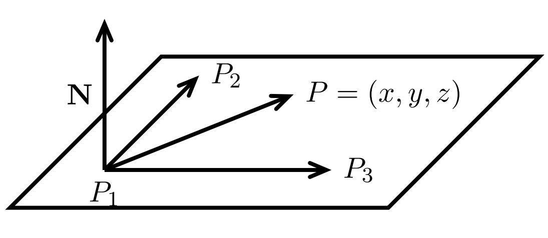
Example 1: Find the plane through the point (1,4,9) with normal $\langle 2, 3, 4\rangle$.

<u>Answer:</u> Point-normal form of the plane is $2(x-1)+3(y-4)+4(z-9) = 0$. We can also write this as $2x + 3y + 4z = 50$.

Example 2: Find the plane containing the points $P_1 = (1, 2, 3)$, $P_2 = (0, 0, 3)$, $P_3 = (2, 5, 5)$.

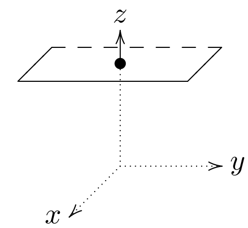
<u>Answer:</u> The goal is to find the basic data, i.e. a point in the plane and a normal to the plane. The point is easy, we already have three of them. To get the normal we note (see figure below) that $\overrightarrow{\mathbf{P_1P_2}}$ and $\overrightarrow{\mathbf{P_1P_3}}$ are vectors in the plane, so their cross product is orthogonal (normal) to the plane. That is,

$$\mathbf{N} = \overrightarrow{\mathbf{P_1P_2}} \times \overrightarrow{\mathbf{P_1P_3}} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 0 \\ 1 & 3 & 2 \end{pmatrix} = -4\mathbf{i} - \mathbf{j}(-2) + \mathbf{k}(-1) = \langle -4, 2, -1\rangle.$$



Using point-normal form (with point P_1) the equation of the plane is

$$-4(x-1)+2(y-2)-(z-3)=0, \text{ or equivalently } \quad -4x+2y-z=-3.$$



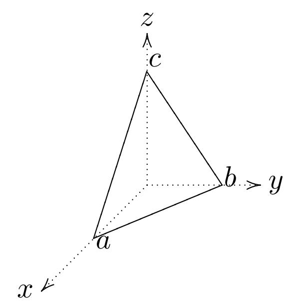
Example 3: Find the plane with normal $\mathbf{N}=\widehat{\mathbf{k}}$ containing the point (0,0,3)
Eq. of plane: $\langle 0,0,1\rangle\cdot\langle x,y,z-3\rangle=0 \;\Leftrightarrow\; z=3$.

Example 4: Find the plane with x, y and z intercepts a, b and c.
Answer: We could find this using the method example 1. Instead, we'll use a shortcut that works when all the intercepts are known. In this case, the intercepts are

$$(a,0,0), \quad (0,b,0), \quad (0,0,c)$$

and we simply write the plane as

$$x/a+y/b+z/c=1.$$



You can easily check that each of the given points is on the plane.
For completeness we'll do this using the more general method of example 1.
The 3 points give us 2 vectors in the plane, $\langle -a,b,0\rangle$ and $\langle -a,0,c\rangle$.
$\Rightarrow\; \mathbf{N}=\langle -a,b,0\rangle\times\langle -a,0,c\rangle=\langle bc,ac,ab\rangle$.
Point-normal form: $bc(x-a)+ac(y-0)+ab(z-0)=0$
$\Leftrightarrow\; bc\,x+ac\,y+ab\,z=abc \;\Leftrightarrow\; x/a+y/b+z/c=1$.

Lines in the plane

While we're at it, let's look at two ways to write the equation of a line in the xy-plane.

Slope-intercept form: Given the slope m and the y-intercept b the equation of a line can be written $y=mx+b$.

Point-normal form:

We can also use point-normal form to find the equation of a line.

Given a point (x_0,y_0) on the line and a vector $\langle a,b\rangle$ normal to the line the equation of the line can be written $a(x-x_0)+b(y-y_0)=0$.