Lagrange Multipliers

We will give the argument for why Lagrange multipliers work later. Here, we'll look at where and how to use them. Lagrange multipliers are used to solve constrained optimization problems. That is, suppose you have a function, say $f(x, y)$, for which you want to find the maximum or minimum value. But, you are not allowed to consider all (x, y) while you look for this value. Instead, the (x, y) you can consider are constrained to lie on some curve or surface. There are lots of examples of this in science, engineering and economics, for example, optimizing some utility function under budget constraints.

Lagrange multipliers problem:
Minimize (or maximize) $w = f(x, y, z)$ constrained by $g(x, y, z) = c$.

Lagrange multipliers solution:
Local minima (or maxima) must occur at a *critical point*. This is a point where $\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g$, and $g(x, y, z) = c$.

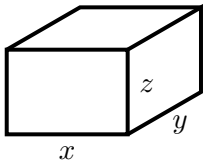
Example: Making a box using a minimum amount of material.

A box is made of cardboard with double thick sides, a triple thick bottom, single thick front and back and no top. It's volume is fixed at 3.

What dimensions use the least amount of cardboard?

Answer: We did this problem once before by solving for z in terms of x and y and substituting for it. That led to an unconstrained optimization problem in x and y. Here we will do it as a constrained problem. It is important to be able to do this because eliminating one variable is not always easy.

The box shown has dimensions x, y, and z.



The area of one side $= yz$. There are two double thick sides $\Rightarrow$ cardboard used $= 4yz$.

The area of the front (and back) $= xz$. It is single thick $\Rightarrow$ cardboard used $= 2xz$.

The area of the bottom $= xy$. It is triple thick $\Rightarrow$ cardboard used $= 3xy$.

Thus, the total cardboard used is

$$w = f(x, y, z) = 4yz + 2xz + 3xy.$$

The fixed volume acts as the constraint. It forces a relation between x, y and z so they can't all be varied independently. The constraint is

$$V = xyz = 3.$$

Our first job is to set up the equations to look for critical points. $\boldsymbol{\nabla} f = \langle 2z + 3y, 4z + 3x, 4y + 2x \rangle$ and $\boldsymbol{\nabla} V = \langle yz, xz, xy \rangle$.

The Lagrange multiplier equations are then

$$\begin{aligned} & \boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} V, \text{ and } V = 3 \\ \Leftrightarrow \quad & \langle 2z + 3y,\, 4z + 3x,\, 4y + 2x \rangle = \lambda \langle yz,\, xz,\, xy \rangle, \quad xyz = 3 \end{aligned}$$

Next we solve these equations for critical points. We do this by solving for λ in each equation (we call this *solving symmetrically*).

$\frac{2z+3y}{yz} = \lambda \quad \frac{4z+3x}{xz} = \lambda, \quad \frac{4y+2x}{xy} = \lambda, \quad xyz = 3 \quad \Rightarrow \ \frac{2}{y} + \frac{3}{z} = \frac{4}{x} + \frac{3}{z} = \frac{4}{x} + \frac{2}{y}$

$\Rightarrow \ \frac{2}{y} = \frac{4}{x} \ \Rightarrow \ x = 2y \quad \text{and} \quad \frac{3}{z} = \frac{2}{y} \ \Rightarrow \ z = \frac{3}{2}y$

Now, $\ xyz = 3 \ \Rightarrow \ 3y^3 = 3 \ \Rightarrow \ y = 1$

Answer: $x = 2, \quad y = 1, \quad z = \frac{3}{2}, \quad w = 18.$

Sphere example:

Minimize $w = y$ constrained to $x^2 + y^2 + z^2 = 1$.

Answer: $\nabla f = \langle 0,\, 1,\, 0\rangle$, $\nabla g = \langle 2x,\, 2y,\, 2z\rangle$

$\nabla f = \lambda \nabla g \ \Rightarrow \ \langle 0,\, 1,\, 0\rangle = \lambda\langle 2x,\, 2y,\, 2z\rangle \ \Rightarrow \ x = z = 0.$

Constraint $\Rightarrow \ y = \pm 1$. (Gives the minimum and maximum respectively).

Example: (**checking the boundary**)

A rectangle in the plane is placed in the first quadrant so that one corner O is at the origin and the two sides adjacent to O are on the axes. The corner P opposite O is on the curve $x + 2y = 1$. Using Lagrange multipliers find for which point P the rectangle has maximum area. Say how you know this point gives the maximum.

Answer: We need some names

$g(x, y) = x + 2y = 1 =$ the constraint and $f(x, y) = xy =$ the area.

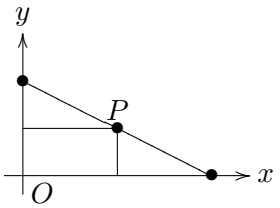
The gradients are: $\nabla g = \widehat{\mathbf{i}} + 2\widehat{\mathbf{j}}, \quad \nabla f = y\,\widehat{\mathbf{i}} + x\,\widehat{\mathbf{j}}.$

Lagrange multipliers: $\Rightarrow \ y = \lambda, \quad x = 2\lambda, \quad x + 2y = 1.$

The first two equations $\Rightarrow \ x = 2y;$

Combine this with the third equation $\Rightarrow \ 4y = 1.$

$\Rightarrow \ y = 1/4, \quad x = 1/2 \ \Rightarrow \ P = (1/2, 1/4).$



We know this is a maximum because the maximum occurs either at a critical point or on the boundary. In this case, the boundary points are on the axes at (1,0) and (0,1/2), which gives a rectangle with area = 0.

Example: (**boundary at** ∞)

A rectangle in the plane is placed in the first quadrant so that one corner O is at the origin and the two sides adjacent to O are on the axes. The corner P opposite O is on the curve $xy = 1$. Using Lagrange multipliers find for which point P the rectangle has minimum perimeter. Say how you know this point gives the minimum.

Answer: Let $g(x, y) = xy = 1 =$ the constraint and $f(x, y) = 2x + 2y =$ the perimeter.

Gradients: $\nabla g = y\,\widehat{\mathbf{i}} + x\,\widehat{\mathbf{j}}, \quad \nabla f = 2\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}}.$

Lagrange multipliers: $\Rightarrow \quad 2 = \lambda y$

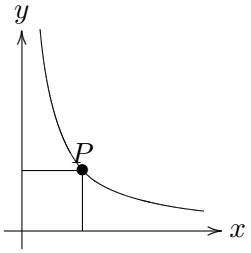
$2 = \lambda x$

$xy = 1$

The first two equations $\Rightarrow \ x = y;$

Combine this with the third equation $\Rightarrow \ x^2 = 1.$

$\Rightarrow \ x = 1, \quad x = 1 \ \Rightarrow \ P = (1, 1).$



We know this is a minimum because the minimum occurs either at a critical point or on the boundary. In this case the boundary points are infinitely far out on the axes which gives a rectangle with perimeter = ∞.

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