V4.3 Physical meaning of curl

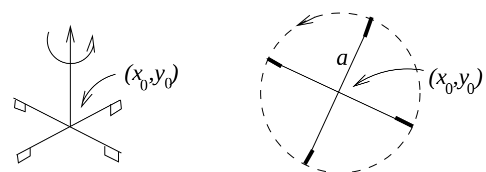
3. An interpretation for curl F.

We will start by looking at the two dimensional curl in the xy-plane. Our interpretation will be that the curl at a point represents twice the angular velocity of a small paddle wheel at that point. At the very end we will indicate how to extend this interpretation to 3 dimensions.

The function curl $\mathbf{F}$ can be thought of as measuring the tendency of $\mathbf{F}$ to produce rotation. Interpreting $\mathbf{F}$ either as a force field or a velocity field, $\mathbf{F}$ will make a suitable test object placed at a point P_0 spin about a vertical axis (i.e., one in the $\mathbf{k}$-direction), and the angular velocity of the spin will be proportional to $(\text{curl } \mathbf{F})_0$.

To see this for the velocity field $\mathbf{v}$ of a flowing liquid, place a paddle wheel of radius a so its center is at (x_0, y_0), and its axis is vertical. We ask how rapidly the flow spins the wheel.

If the wheel had only one blade, the velocity of the blade would be $\mathbf{F}\cdot\mathbf{t}$, the component of the flow velocity vector $\mathbf{F}$ perpendicular to the blade, i.e., tangent to the circle of radius a traced out by the blade.



paddlewheel   top view

Since $\mathbf{F}\cdot\mathbf{t}$ is not constant along this circle, if the wheel had only one blade it would spin around at an uneven rate. But if the wheel has many blades, this unevenness will be averaged out, and it will spin around at approximately the *average value* of the tangential velocity $\mathbf{F}\cdot\mathbf{t}$ over the circle. Like the average value of any function defined along a curve, this average tangential velocity can be found by integrating $\mathbf{F}\cdot\mathbf{t}$ over the circle, and dividing by the length of the circle. Thus,

$$\begin{aligned}\text{speed of blade} &= \frac{1}{2\pi a}\oint_C \mathbf{F}\cdot\mathbf{t}\,ds \quad = \frac{1}{2\pi a}\oint_C \mathbf{F}\cdot d\mathbf{r} \\ &= \frac{1}{2\pi a}\iint_R (\text{curl } \mathbf{F})_0\,dx\,dy, \qquad \text{by Green's theorem,} \\ &\approx \frac{1}{2\pi a}(\text{curl } \mathbf{F})_0 \pi a^2\ , \qquad (8)\end{aligned}$$

where $(\text{curl } \mathbf{F})_0$ is the value of the function curl $\mathbf{F}$ at (x_0, y_0). The justification for the last approximation is that if the circle formed by the paddlewheel is small, then curl $\mathbf{F}$ has approximately the value $(\text{curl } \mathbf{F})_0$ over the interior R of the circle, so that multiplying this constant value by the area πa^2 of R should give approximately the value of the double integral.

From (8) we get for the tangential speed of the paddlewheel:

$$\text{tangential speed} \quad \approx \quad \frac{a}{2}(\text{curl } \mathbf{F})_0\ . \qquad (9)$$

We can get rid of the a by using the angular velocity ω_0 of the paddlewheel; since the tangential speed is $a\omega_0$, (9) becomes

$$\omega_0 \quad \approx \quad \frac{1}{2}(\text{curl } \mathbf{F})_0\ . \qquad (10)$$

As the radius of the paddlewheel gets smaller, the approximation becomes more exact, and passing to the limit as $a \to 0$, we conclude that, for a two-dimensional velocity field $\mathbf{F}$,

$$(11) \qquad \boxed{\text{curl } \mathbf{F} = \textit{twice the angular velocity of an infinitesimal paddlewheel at } (x, y)\ .}$$

The curl thus measures the "vorticity" of the fluid flow — its tendency to produce rotation.

A consideration of curl $\mathbf{F}$ for a force field would be similar, interpreting $\mathbf{F}$ as exerting a torque on a spinnable object — a little dumbbell with two unit masses for a gravitational field, or with two unit positive charges for an electrostatic force field.

Example 1. Calculate and interpret curl $\mathbf{F}$ for (a) $x\,\mathbf{i} + y\,\mathbf{j}$ (b) $\omega(-y\,\mathbf{i} + x\,\mathbf{j})$

Solution. (a) curl $\mathbf{F} = 0$; this makes sense since the field is radially outward and radially symmetric, there is no favored angular direction in which the paddlewheel could spin.

(b) curl $\mathbf{F} = 2\omega$ at every point. Since this field represents a fluid rotating about the origin with constant angular velocity ω (see section V1), it is at least clear that curl $\mathbf{F}$ should be 2ω at the origin; it's not so clear that it should have this same value everywhere, but it is true.

Extension to Three Dimensions. To extend this interpretation to three dimensions note that any component of the flow of $\mathbf{F}$ in the $\mathbf{k}$ direction will not have any effect on a paddle wheel in the xy-plane. In fact, for any plane with normal $\mathbf{n}$ the component of $\mathbf{F}$ in the direction of $\mathbf{n}$ has no effect on a paddle wheel in the plane. This leads to the following interpretation of the three dimensional curl:

For any plane with unit normal $\mathbf{n}$, $(\text{curl } \mathbf{F}) \cdot \mathbf{n}$ is two times the angular velocity of a small paddle wheel in the plane.

We could force through a proof along the lines of the 2D proof above. Once we learn Stokes Theorem we can make a much simpler argument.