Kepler's Second Law

By studying the Danish astronomer Tycho Brahe's data about the motion of the planets, Kepler formulated three empirical laws; two of them can be stated as follows:

Second Law A planet moves in a plane, and the radius vector (from the sun to the planet) sweeps out equal areas in equal times.

First Law The planet's orbit in that plane is an ellipse, with the sun at one focus.

From these laws, Newton deduced that the force keeping the planets in their orbits had magnitude $1/d^2$, where d was the distance to the sun; moreover, it was directed toward the sun, or as was said, *central*, since the sun was placed at the origin.

Using a little vector analysis (without coordinates), this section is devoted to showing that *the Second Law is equivalent to the force being central.*

> It is harder to show that an elliptical orbit implies the magnitude of the force is of the form K/d^2, and vice-versa; this uses vector analysis in polar coordinates and requires the solution of non-linear differential equations.

1. Differentiation of products of vectors

Let $\mathbf{r}(t)$ and $\mathbf{s}(t)$ be two differentiable vector functions in 2- or 3-space. Then

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) \;=\; \frac{d\mathbf{r}}{dt}\cdot\mathbf{s} \;+\; \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}; \qquad \frac{d}{dt}(\mathbf{r}\times\mathbf{s}) \;=\; \frac{d\mathbf{r}}{dt}\times\mathbf{s} \;+\; \mathbf{r}\times\frac{d\mathbf{s}}{dt}. \tag{1}$$

These rules are just like the product rule for differentiation. Be careful in the second rule to get the multiplication order correct on the right, since $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}$ in general. The two rules can be proved by writing everything out in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components and differentiating. They can also be proved directly from the definition of derivative, without resorting to components, as follows:

Let t increase by Δt. Then $\mathbf{r}$ increases by $\Delta\mathbf{r}$, and $\mathbf{s}$ by $\Delta\mathbf{s}$, and the corresponding change in $\mathbf{r}\cdot\mathbf{s}$ is given by

$$\Delta(\mathbf{r}\cdot\mathbf{s}) \;=\; (\mathbf{r}+\Delta\mathbf{r})\cdot(\mathbf{s}+\Delta\mathbf{s}) \;-\; \mathbf{r}\cdot\mathbf{s}\ ,$$

so if we expand the right side out and divide all terms by Δt, we get

$$\frac{\Delta(\mathbf{r}\cdot\mathbf{s})}{\Delta t} \;=\; \frac{\Delta\mathbf{r}}{\Delta t}\cdot\mathbf{s} \;+\; \mathbf{r}\cdot\frac{\Delta\mathbf{s}}{\Delta t} \;+\; \frac{\Delta\mathbf{r}}{\Delta t}\cdot\Delta\mathbf{s}\ .$$

Now let $\Delta t\to 0$; then $\Delta\mathbf{s}\to 0$ since $\mathbf{s}(t)$ is continuous, and we get the first equation in (1). The second equation in (1) is proved the same way, replacing $\cdot$ by $\times$ everywhere.

2. Kepler's second law and the central force. To show that the force being central (i.e., directed toward the sun) is equivalent to Kepler's second law, we need to translate that law into calculus. "Sweeps out equal areas in equal times" means:

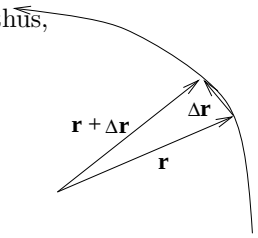
the radius vector sweeps out area at a constant rate .

The first thing therefore is to obtain a mathematical expression for this rate. Referring to the picture, we see that as the time increases from t to $t+\Delta t$, the corresponding change in the area A is given approximately by

$$\Delta A \;\approx\; \text{area of the triangle} \;=\; \frac{1}{2}\,|\mathbf{r}\times\Delta\mathbf{r}|\ ,$$

since the triangle has half the area of the parallelogram formed by $\mathbf{r}$ and $\Delta\mathbf{r}$; thus,

$$2\,\frac{\Delta A}{\Delta t} \;\approx\; \left|\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t}\right|,$$



and as $\Delta t\to 0$, this becomes

$$2\,\frac{dA}{dt} \;=\; \left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right| \;=\; |\mathbf{r}\times\mathbf{v}|. \qquad \text{where } \mathbf{v}=\frac{d\mathbf{r}}{dt}\ . \tag{2}$$

Using (2), we can interpret Kepler's second law mathematically. Since the area is swept out at a constant rate, dA/dt is constant, so according to (2),

$$|\mathbf{r}\times\mathbf{v}| \quad \textit{is a constant.} \tag{3}$$

Moreover, since Kepler's law says $\mathbf{r}$ lies in a plane, the velocity vector $\mathbf{v}$ also lies in the same plane, and therefore

$$\mathbf{r}\times\mathbf{v} \quad \textit{has constant direction (perpendicular to the plane of motion).} \tag{4}$$

Since the direction and magnitude of $\mathbf{r}\times\mathbf{v}$ are both constant,

$$\mathbf{r}\times\mathbf{v} \;=\; \mathbf{K}, \text{ a constant vector,} \tag{5}$$

and from this we see that

$$\frac{d}{dt}\,(\mathbf{r}\times\mathbf{v}) \;=\; \mathbf{0}\ . \tag{6}$$

But according to the rule (1) for differentiating a vector product,

$$\begin{aligned} \frac{d}{dt}\,(\mathbf{r}\times\mathbf{v}) \;&=\; \mathbf{v}\times\mathbf{v}+\mathbf{r}\times\mathbf{a}, && \text{where } \mathbf{a}=\frac{d\mathbf{v}}{dt}, \\ &=\; \mathbf{r}\times\mathbf{a}, && \text{since } \mathbf{s}\times\mathbf{s}=\mathbf{0} \text{ for any vector } \mathbf{s}. \end{aligned} \tag{7}$$

Now (6) and (7) together imply

$$\mathbf{r}\times\mathbf{a} \;=\; \mathbf{0}, \tag{8}$$

which shows that *the acceleration vector* $\mathbf{a}$ *is parallel to* $\mathbf{r}$, but in the opposite direction, since the planets do go around the sun, not shoot off to infinity.

Thus $\mathbf{a}$ is directed toward the center (i.e., the sun), and since $\mathbf{F}=m\mathbf{a}$, the force $\mathbf{F}$ is also directed toward the sun. (Note that "center" does not mean the center of the elliptical orbits, but the mathematical origin, i.e., the tail of the radius vector $\mathbf{r}$, which we are taking to be the sun's position.)

The reasoning is reversible, so for motion under any type of central force, the path of motion will lie in a plane and area will be swept out by the radius vector at a constant rate.

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